

Automorphism groups of edge-transitive maps

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Abstract

For each of the 14 classes of edge-transitive maps described by Graver and Watkins, necessary and sufficient conditions are given for a group to be the automorphism group of a map, or of an orientable map without boundary, in that class. Extending earlier results of Širáň, Tucker and Watkins, these are used to determine which symmetric groups S_n can arise in this way for each class. Similar results are obtained for all finite simple groups, building on work of Leemans and Liebeck, Nuzhin and others on generating sets for such groups. It is also shown that each edge-transitive class realises finite groups of every sufficiently large nilpotence class or derived length, and also realises uncountably many non-isomorphic infinite groups.

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Key words: Edge-transitive map, regular map, chiral map, automorphism group, finite simple group.

1 Introduction

Two of the most important and deeply studied classes of maps on surfaces are those consisting of the regular maps and of the orientably regular chiral maps (here simply called chiral maps for brevity). The regular maps \mathcal{M} are those for which the automorphism group $\text{Aut } \mathcal{M}$ acts transitively on flags,

while the chiral maps are those orientable maps for which $\text{Aut } \mathcal{M}$ is transitive on arcs but not flags, so that \mathcal{M} is not isomorphic to its mirror image.

Such maps are all edge-transitive. In 1997 Graver and Watkins [14] partitioned edge-transitive maps \mathcal{M} into 14 classes, distinguished by the isomorphism class of the quotient map $\mathcal{M}/\text{Aut } \mathcal{M}$ (see §4.2 for a summary of this classification). These classes T correspond bijectively to the 14 isomorphism classes of maps $\mathcal{N}(T)$ with one edge; they include class 1, consisting of the regular maps, and class 2^Pex , consisting of the chiral maps, together with others such as class 3, consisting of the just-edge-transitive maps, those for which $\text{Aut } \mathcal{M}$ is transitive on edges but not on vertices or faces. The main object of this paper is to study, for each of these classes T , the set $\mathcal{G}(T)$ of groups which can be realised as the automorphism group of a map in T .

It is widely known, and not hard to prove, that apart from a few small exceptions each finite symmetric or alternating group can be realised as the automorphism group of a regular map, and that the same applies to chiral maps. In 2001 Širáň, Tucker and Watkins [47] showed that for each integer $n \geq 11$ with $n \equiv 3$ or $11 \pmod{12}$, there are finite, orientable, edge-transitive maps \mathcal{M} in each of the 14 classes, with $\text{Aut } \mathcal{M}$ isomorphic to the symmetric group S_n . The first aim of this paper is to extend their result by determining, for each class T , all integers n such that $S_n \in \mathcal{G}(T)$. This result, together with similar results for the alternating groups A_n and the projective special linear groups $L_2(q) = \text{PSL}_2(q)$, is summarised in Theorem 1.1 and Table 1:

Theorem 1.1 *A symmetric group S_n , an alternating group A_n , or a projective special linear group $L_2(q)$ is isomorphic to the automorphism group of an edge-transitive map in a class T if and only if it satisfies the corresponding condition in Table 1.*

Class T	S_n	A_n	$L_2(q)$
1	$n \geq 1$	$n = 1, 2, 5$ or $n \geq 9$	$q \neq 3, 7, 9$
2, 2^* , 2^P	$n \geq 2$	$n \geq 5$	$q \neq 3$
2ex , 2^*ex , 2^Pex	$n \geq 6$	$n \geq 8$	no q
3	$n \geq 2$	$n \geq 5$	$q \neq 3$
4, 4^* , 4^P	$n \geq 2$	$n \geq 4$	every q
5, 5^* , 5^P	$n \geq 6$	$n \geq 7$	no q

Table 1: Groups S_n , A_n and $L_2(q)$ in sets $\mathcal{G}(T)$.

In most cases, all but finitely many of these groups are realised in each class T , the exceptions being six of the classes for which no groups $L_2(q)$ arise. The method of proof, both here and for other results stated below, is to follow [47] in using necessary and sufficient conditions for a group to be in the various sets $\mathcal{G}(T)$: for instance, when $T = 1$ these require the group to have generators r_i ($i = 0, 1, 2$) satisfying $r_i^2 = (r_0 r_2)^2 = 1$. In the cases $T \neq 1$ there are similar conditions on generators, together with the requirement that the group should not have certain ‘forbidden automorphisms’ (see §4.5 for details). It is then a routine matter to apply these conditions to the groups in Theorem 1.1.

The groups A_n are simple for all $n \geq 5$, as are the groups $L_2(q)$ for all prime powers $q \geq 4$. More generally, it is of interest to determine, for each class T , which non-abelian finite simple groups are in $\mathcal{G}(T)$. In fact, it is easier to list those which are not, as in Table 2 (where we use ATLAS notation [6] for simple groups):

Theorem 1.2 *A non-abelian finite simple group is isomorphic to the automorphism group of an edge-transitive map in a class T if and only if it is not one of the exceptions listed in the corresponding row of Table 2.*

Class T	Non-abelian finite simple groups $G \notin \mathcal{G}(T)$
1	$L_3(q), U_3(q), L_4(2^e), U_4(2^e), U_4(3), U_5(2),$ $A_6, A_7, M_{11}, M_{22}, M_{23}, McL$
$2, 2^*, 2^P$	$U_3(3)$
$2 \text{ ex}, 2^* \text{ ex}, 2^P \text{ ex}$	$L_2(q), L_3(q), U_3(q), A_7$
3	—
$4, 4^*, 4^P$	—
$5, 5^*, 5^P$	$L_2(q)$

Table 2: Non-abelian finite simple groups not in sets $\mathcal{G}(T)$.

Here M_n is the Mathieu group of degree n , while McL is the McLaughlin group. In these lists, all integers $e \geq 1$ are allowed, as are all prime powers q provided the corresponding group is simple. Note that the exceptions listed in the first row include the groups $L_2(7) \cong L_3(2)$, $L_2(9) \cong A_6$ and $A_8 \cong L_4(2)$, while those in the third row include $A_5 \cong L_2(4) \cong L_2(5)$ and $A_6 \cong L_2(9)$.

The entry for $T = 1$, the class of regular maps, is due to Nuzhin and others, through their answer to Mazurov’s (purely algebraic) question [28,

Problem 7.30] asking which non-abelian finite simple groups can be generated by three involutions, two of them commuting (see Section 9 for details, including the addition of $U_4(3)$ and $U_5(2)$ to previously published lists). The solution for $T = 2^P \text{ex}$, the class of chiral maps, has recently been determined by Leemans and Liebeck [30] in the equivalent context of abstract polyhedra (again, see Section 9), and a simple argument using map dualities extends their result to the classes 2ex and 2^*ex . The entries for the remaining ten classes are apparently new, and the proofs are given in Section 9.

The exceptions for these ten classes are easily explained. The unitary group $U_3(3)$ is not in $\mathcal{G}(T)$ for the classes $T = 2, 2^*$ and 2^P since groups realised in such classes must be generated by at most three involutions, and Wagner [49] has shown that this group requires four. By an observation of Singerman [46], for each generating pair for $L_2(q)$ there is an automorphism inverting both generators; such an automorphism is forbidden for the classes $T = 5, 5^*$ and 5^P , so $L_2(q) \notin \mathcal{G}(T)$. The exceptions for $T = 2 \text{ex}$ and 2^*ex are the same as those found in [30] for $T = 2^P \text{ex}$.

The main part of the proofs of these theorems consists of showing that various groups are realised (as an automorphism group) in particular classes. Simple arguments show that if any group is realised in class 1 or 2^Pex then it is also realised in various other classes (see Lemma 4.4 for a precise statement), so this allows one to concentrate on those groups which are not in $\mathcal{G}(1)$ or not in $\mathcal{G}(2^P \text{ex})$ (in the case of finite simple groups, these are the exceptions in the first and third rows of Table 2). In order to realise such groups, more direct arguments are required, finding specific generators and then showing that these do not admit forbidden automorphisms.

In this paper we also consider the set $\mathcal{G}^+(T)$ of groups which can be ‘evenly realised’, that is, as the automorphism group of an orientable map without boundary, in each of the classes T . If $T = 2^P \text{ex}, 5$ or 5^* then all maps in that class have these properties, so $\mathcal{G}^+(T) = \mathcal{G}(T)$. For the other eleven classes, each group in $\mathcal{G}^+(T)$ must have a subgroup of index 2, so in particular no simple groups (other than C_2 for $T = 1$) can be evenly realised in such a class T . For example, it follows from Theorem 1.1 that no group $L_2(q)$ is evenly realised by an edge-transitive map. In the case of the groups S_n and A_n we have the following:

Theorem 1.3 *A symmetric group S_n or an alternating group A_n is isomorphic to the automorphism group of an edge-transitive orientable map without boundary in a class T if and only if it satisfies the corresponding condition*

in Table 3.

Class T	$S_n \in \mathcal{G}^+(T)$	$A_n \in \mathcal{G}^+(T)$
1	$n \neq 1, 5, 6$	no n
2, 2^*	$n \neq 1, 2, 5, 6$	no n
2^P	$n \geq 3$	no n
2 ex, 2^* ex	$n \geq 7$	no n
2^P ex	$n \geq 6$	$n \geq 8$
3	$n \geq 3$	no n
4, 4^* , 4^P	$n \geq 3$	no n
5, 5^*	$n \geq 6$	$n \geq 7$
5^P	$n \geq 6$	no n

Table 3: Groups S_n and A_n in sets $\mathcal{G}^+(T)$.

With a few small exceptions, the automorphism groups considered above are all non-solvable. The following result shows that each class also realises finite groups of every sufficiently large nilpotence class or derived length:

Theorem 1.4 *There is a finite group of nilpotence class c , or of derived length l , isomorphic to the automorphism group of an edge-transitive map in a class T , if and only if c or l satisfy the corresponding condition in Table 4.*

Class T	Nilpotence class c	Derived length l
2 ex, 2^* ex, 2^P ex	$c \geq 5$	$l \geq 2$
5, 5^* , 5^P	$c \geq 2$	$l \geq 2$
All other T	$c \geq 1$	$l \geq 1$

Table 4: Nilpotence class and derived length.

Non-compact edge-transitive maps, with infinite automorphism groups, are considered in Section 11. Combinatorial group theory is used to show that each class T realises uncountably many non-isomorphic groups, and that every countable group can be embedded in such a group. Questions of growth and decidability are also considered.

This paper concentrates mainly on group-theoretic properties of edge-transitive maps without boundary. It is hoped to relate these to their topological and combinatorial properties, such as orientability, genus and type, and to consider maps with boundary, in future papers.

2 Outline of the method

The following method is used to prove these results; full details are given in later sections. Maps \mathcal{M} (always assumed to be connected) can be regarded as transitive permutation representations of the group

$$\Gamma = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle,$$

acting as the monodromy group of \mathcal{M} by permuting its flags. This group Γ is the free product of a Klein four group $E = \langle R_0, R_2 \rangle \cong V_4$ and a cyclic group $\langle R_1 \rangle \cong C_2$. Each map \mathcal{M} has automorphism group $\text{Aut } \mathcal{M} \cong N_\Gamma(M)/M$ where M (a *map subgroup*) is the stabiliser in Γ of a flag. Then \mathcal{M} is edge-transitive if and only if $\Gamma = N_\Gamma(M)E$. The 14 classes T of edge-transitive maps correspond to the 14 conjugacy classes of subgroups $N = N(T) \leq \Gamma$ satisfying $\Gamma = NE$: the maps \mathcal{M} in each class are those for which M has normaliser N , so that $\text{Aut } \mathcal{M} \cong N/M$ and \mathcal{M} is a regular covering of the single-edge map $\mathcal{N}(T)$ with map subgroup N . For instance $N(1) = \Gamma$, so that regular maps correspond to normal subgroups M of Γ , while $N(2^{\text{Pex}})$ is the even subgroup Γ^+ of index 2 in Γ , consisting of the words of even length in the generators R_i , and $N(3)$ is the normal closure of R_1 .

The Reidemeister-Schreier process yields presentations for these subgroups $N = N(T)$ of Γ (all of index at most 4), and these give information about their quotients. Thus the quotients of Γ (and hence the automorphism groups of regular maps) are those groups generated by three elements of order dividing 2, with two of them commuting; for instance, as a result of work of Nuzhin and others [37, 39, 40, 41, 42], it is known which finite simple groups have this property (see §9 for details). However, if $T \neq 1$ then not all quotients N/M of N are in $\mathcal{G}(T)$: we also require N/M not to have certain ‘forbidden automorphisms’ (see Table 5 in §4.5, also [47, Condition 3.2]) which would cause $N_\Gamma(M)$ to be strictly larger than N . For instance, $N(2^{\text{Pex}}) = \Gamma^+ = \langle X, Y \mid Y^2 = 1 \rangle \cong C_\infty * C_2$ where $X = R_1 R_2$ and $Y = R_0 R_2$, and the corresponding quotients, the groups in $\mathcal{G}(2^{\text{Pex}})$, are those with no automorphism inverting the images of X and Y ; Leemans and Liebeck [29, 30] have recently characterised the finite simple groups with this property (again, see §9 for details).

The task of determining such quotients for each class T is eased by using a group Ω of map operations, introduced by Wilson [52] and generated by the classical and Petrie dualities for maps: induced by the outer automorphism group $\text{Out } \Gamma \cong \text{Aut } E \cong S_3$ of Γ acting on the conjugacy classes of subgroups

N , this group permutes the 14 classes T in six orbits, corresponding to the six rows in Tables 1 and 2; since Ω preserves the sets of automorphism groups realised by these classes, it is sufficient to consider one representative class from each orbit. For this we will choose the classes $T = 1, 2, 3, 4$ and 5, though in the case of the remaining orbit it is often more convenient to choose the class 2^Pex of chiral maps, rather than $T = 2\text{ex}$.

These groups $N(T)$ have decompositions as free products, namely

$$N(1) = \Gamma \cong V_4 * C_2, \quad N(2) \cong C_2 * C_2 * C_2, \quad N(2^P\text{ex}) \cong C_\infty * C_2,$$

$$N(3) \cong C_2 * C_2 * C_2 * C_2, \quad N(4) \cong C_2 * C_2 * C_\infty, \quad N(5) \cong C_\infty * C_\infty \cong F_2,$$

leading to further simplifications. There are epimorphisms $N(T) \rightarrow \Gamma$ for $T = 2, 3$ and 4, so that every quotient G of Γ is also a quotient of $N(T)$; these epimorphisms can be chosen so that in almost all cases G has no forbidden automorphisms, and is therefore realised in classes $T = 2, 3$ and 4. Similarly, an obvious epimorphism $N(5) \rightarrow N(2^P\text{ex})$ means that any group in $\mathcal{G}(2^P\text{ex})$ is also in $\mathcal{G}(5)$. Although it does not solve the realisation problem completely, this simplification (stated more precisely in Lemma 4.4) means that a large part of the solution can be obtained by concentrating on the two groups Γ and Γ^+ and their quotients, namely the automorphism groups of the regular maps and the orientably regular maps. Fortunately there is a significant amount of useful material already in the literature, or in accessible databases, about these groups. For example, a series of papers by Nuzhin and others characterises the non-abelian finite simple groups in $\mathcal{G}(1)$, while a recent result of Leemans and Liebeck [30] does the same for $\mathcal{G}(2^P\text{ex})$.

Nevertheless, there are cases not covered by these simplifications; for these one has to find explicit generating sets satisfying the relevant conditions for each class T , or (in a few exceptional cases) to prove that no such sets exist. In the case of the alternating and symmetric groups, we use classical and more recent results on their generators. For the simple groups of Lie type and the sporadic simple groups, we use results and techniques of Malle, Saxl and Weigel [35], together with the detailed information in the ATLAS [6] concerning automorphisms, maximal subgroups, conjugacy classes and character tables.

3 Algebraic theory of maps

Here we briefly outline the algebraic theory of maps developed in more detail elsewhere (see [25], for example, and see [17] for background in topological graph theory).

Each map \mathcal{M} (possibly non-orientable or with non-empty boundary) determines a permutation representation of the group

$$\Gamma = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle \cong V_4 * C_2,$$

on the set Φ of flags $\phi = (v, e, f)$ of \mathcal{M} , where v, e and f are a mutually incident vertex, edge and face. For each $\phi \in \Phi$ and each $i = 0, 1, 2$, there is at most one flag $\phi' \neq \phi$ with the same j -dimensional components as ϕ for each $j \neq i$ (possibly none if ϕ is a boundary flag). Define r_i to be the permutation of Φ transposing each ϕ with ϕ' if the latter exists, and fixing ϕ otherwise. (See Figures 1 and 2 for the former and latter cases. In Figure 2, as in all diagrams, the broken line represents part of the boundary of the map.). Since $r_i^2 = (r_0 r_2)^2 = 1$ there is a permutation representation

$$\theta : \Gamma \rightarrow G := \langle r_0, r_1, r_2 \rangle \leq \text{Sym } \Phi$$

of Γ on Φ , given by $R_i \mapsto r_i$.

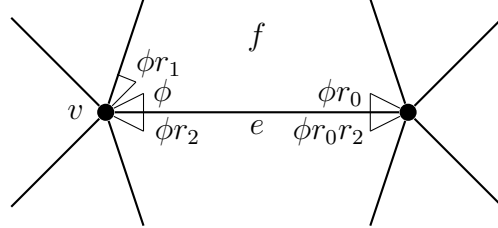


Figure 1: Generators r_i of G acting on a flag $\phi = (v, e, f)$.

Conversely, any permutation representation of Γ on a set Φ determines a map \mathcal{M} in which the vertices, edges and faces are identified with the orbits on Φ of the subgroups $\langle R_1, R_2 \rangle \cong D_\infty$, $\langle R_0, R_2 \rangle \cong V_4$ and $\langle R_0, R_1 \rangle \cong D_\infty$, incident when they have non-empty intersection.

The map \mathcal{M} is connected if and only if Γ acts transitively on Φ , as we will always assume. In this case the stabilisers in Γ of flags $\phi \in \Phi$ form a conjugacy class of subgroups $M \leq \Gamma$, called *map subgroups*.

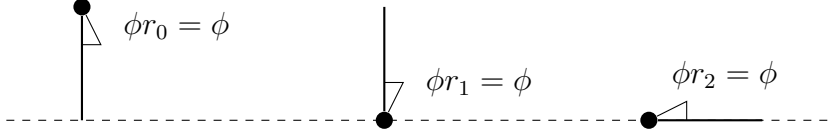


Figure 2: Flags fixed by r_0, r_1 and r_2 .

One can represent Γ as a group of isometries of the hyperbolic plane \mathbb{H} , with the generators R_i acting as reflections in the sides of an ideal right-angled triangle T . This is shown in Figure 3 using the Poincaré disc model of \mathbb{H} , with vertices of T at $0, 1$ and i . The images of T form a tessellation of \mathbb{H} , the barycentric subdivision of the *universal map* \mathcal{M}_∞ , a map in which the vertices and edges are the images of 1 and the unit interval. Then $\mathcal{M} \cong \mathcal{M}_\infty/M$, giving a concrete geometric realisation of \mathcal{M} .

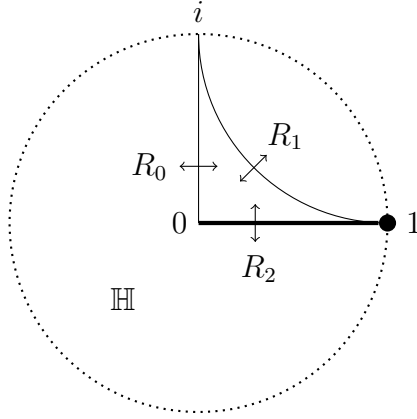


Figure 3: Generators R_i of Γ acting on the hyperbolic plane

The map \mathcal{M} is finite (has finitely many flags) if and only if M has finite index in Γ , and it has non-empty boundary if and only if M contains a conjugate of some R_i , or equivalently some r_i has a fixed point in Φ . In particular, \mathcal{M} is orientable and with empty boundary if and only if M is contained in the even subgroup Γ^+ of index 2 in Γ , consisting of the words of even length in the generators R_i , those preserving the orientation of \mathbb{H} .

The group G is called the *monodromy group* $\text{Mon } \mathcal{M}$ of \mathcal{M} . The *automorphism group* $A = \text{Aut } \mathcal{M}$ of \mathcal{M} is the centraliser of G in $\text{Sym } \Phi$. We

have $A \cong N/M$ where $N := N_\Gamma(M)$ is the normaliser of M in Γ . The map \mathcal{M} is called *regular* if A is transitive on Φ , or equivalently G is a regular permutation group, that is, M is normal in Γ ; in this case

$$A \cong G \cong \Gamma/M,$$

and one can identify Φ with G , so that A and G are the left and right regular representations of G on itself. The map \mathcal{M} is *edge-transitive* if A acts transitively on its edges; this is equivalent to the condition that $\Gamma = NE$, where $E := \langle R_0, R_2 \rangle \cong V_4$.

The (classical) dual $D(\mathcal{M})$ of \mathcal{M} , a map on the same surface formed by transposing the roles of vertices and faces, corresponds to the image of M under the automorphism δ of Γ which fixes R_1 , and transposes R_0 and R_2 . The Petrie dual $P(\mathcal{M})$ embeds the same graph as \mathcal{M} , but the faces are transposed with Petrie polygons, closed zig-zag paths which alternately turn first right and first left at the vertices of \mathcal{M} ; this operation corresponds to the automorphism π of Γ which transposes R_0 and R_0R_2 , and fixes R_1 and R_2 . Both of these operations D and P preserve regularity and automorphism groups, but P may change the underlying surface, for example by changing orientability and by eliminating or introducing boundary components. The group $\Omega = \langle D, P \rangle$ of map operations generated by D and P , introduced by Wilson in [52], is isomorphic to S_3 , permuting vertices, faces and Petrie polygons; it corresponds to the outer automorphism group $\text{Out } \Gamma \cong \text{Aut } E \cong S_3$ of Γ acting on maps by permuting conjugacy classes of map subgroups [25].

4 Edge-transitive maps

Graver and Watkins [14] partitioned edge-transitive maps into 14 classes, mainly for applications to infinite but locally finite edge-transitive planar maps; see their Table 2 for a summary. In this section we will describe and analyse their classification, but from a rather more group-theoretic point of view. This is largely motivated by the work of Širáň, Tucker and Watkins in [47], where finite symmetric groups are used as automorphism groups in order to construct finite maps of particular types in all 14 classes, together with that of Orbanić, Pellicer, Pisanski and Tucker in [43], where edge-transitive maps of small genus are classified. (In the present paper, ‘class’ refers to the 14 equivalence classes of edge-transitive maps, whereas ‘type’ refers to the parameters giving the valencies of the faces and vertices of a map, as in [7].)

4.1 General properties

A map \mathcal{M} , corresponding to a conjugacy class in Γ of map subgroups M , is edge-transitive if and only if $NE = \Gamma$, where $N = N_\Gamma(M)$, and $E := \langle R_0, R_2 \rangle \cong V_4$. There are 14 conjugacy classes of subgroups $N \leq \Gamma$ satisfying $NE = \Gamma$: these are the map subgroups for the 14 isomorphism classes of maps \mathcal{N} with a single edge, shown in Figure 4. The edge-transitive maps \mathcal{M} are thus partitioned into 14 classes T , according to the conjugacy class of $N_\Gamma(M)$, or equivalently the common isomorphism class of the quotient maps

$$\mathcal{N} = \mathcal{N}(T) = \mathcal{M}/\text{Aut } \mathcal{M}$$

of the maps \mathcal{M} in T . We call $\mathcal{N}(T)$ the *basic map* for T , and the group $N = N(T) = N_\Gamma(M)$ (determined only up to conjugacy in Γ) the *parent group* for T , since the maps \mathcal{M} in T are all regular coverings of $\mathcal{N}(T)$ by quotient groups $G \cong N(T)/M$ corresponding to normal subgroups M of $N(T)$. (In essence, $\mathcal{N}(T)$ is a particular case of the symmetry type graph of a polytope or maniplex, studied by Cunningham, del Río-Francos, Hubard and Toledo in [8].)

Note, however, that the map \mathcal{M} corresponding to a normal subgroup M of $N(T)$ need not always be in T : if $N_\Gamma(M)$ properly contains $N(T)$ then $N_\Gamma(M) = N(T')$ for some class $T' \neq T$, and \mathcal{M} is in class T' (see Section 4.4 for details of such pairs T, T'). This happens if and only if \mathcal{M} (or equivalently $N(T)/M$) has certain ‘forbidden automorphisms’ induced by elements of $N(T') \setminus N(T)$, or equivalently by lifting non-identity automorphisms of $\mathcal{N}(T)$ to \mathcal{M} . For example, if we take M to be $N(T)$ itself, so that $\mathcal{M} = \mathcal{N}(T)$, then Figure 4 shows that, except when $T = 1$, \mathcal{M} always has such an automorphism, so that $\mathcal{N}(T)$ is not in class T .

4.2 The 14 classes of edge-transitive maps

The 14 maps \mathcal{N} with one edge are shown in Figure 4, labelled with the symbol $1, 2, 2^*, \dots$, assigned by Graver and Watkins in [14] to the corresponding class T . The underlying surface of each map \mathcal{N} is the closed unit disc $\overline{\mathbb{D}}$, with the following exceptions: when $T = 2^P \text{ex}$, 5 or 5^* it is the 2-sphere S^2 , when $T = 4^P$ it is the Möbius band, and when $T = 5^P$ it is the real projective plane. The letters V, F and P indicate whether \mathcal{N} has one vertex, face or Petrie polygon, and thus whether the maps \mathcal{M} in class T are vertex-, face- or Petrie-transitive. The rows correspond to the orbits of the group Ω of map

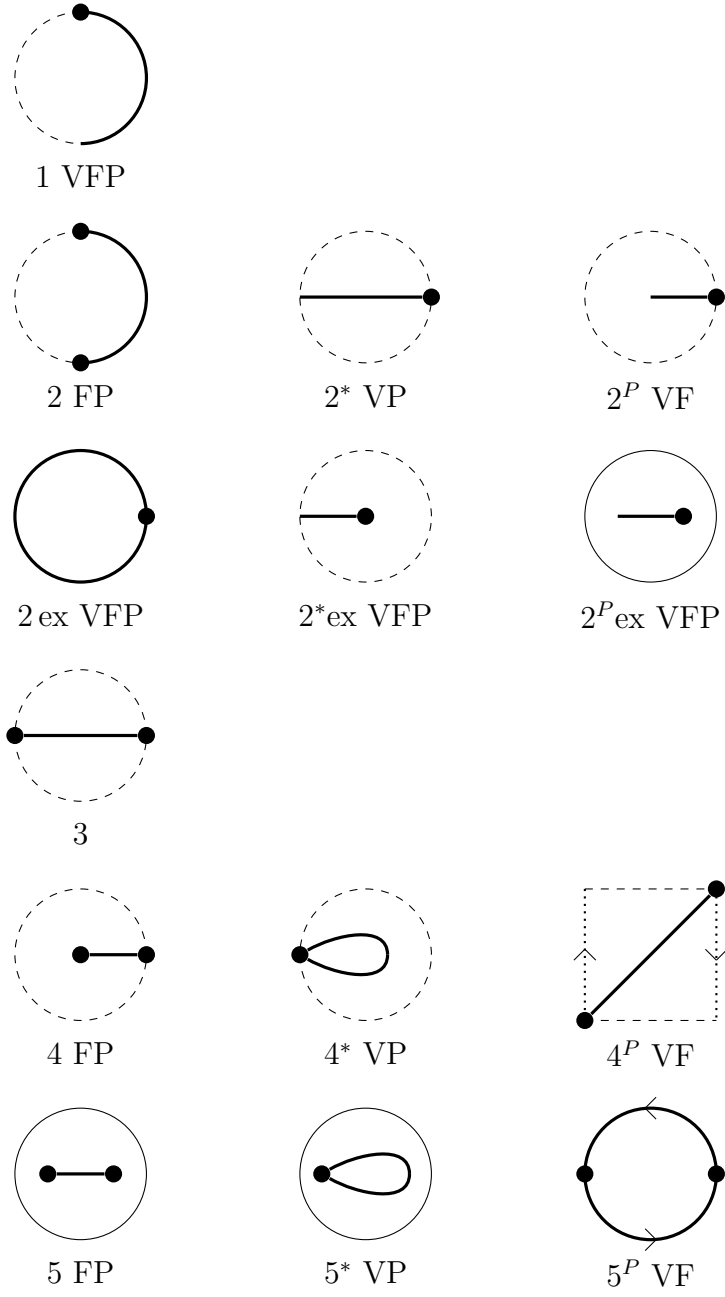


Figure 4: The basic maps \mathcal{N} for the 14 edge-transitive types

operations, and the second and third maps in each row (when they exist) are the dual and the Petrie dual of the first map; the same applies to the maps \mathcal{M} in the corresponding classes.

We will now prove that there are 14 classes T of edge-transitive maps, or equivalently that there are 14 conjugacy classes of subgroups $N = N(T)$ of Γ satisfying $\Gamma = NE$, namely the map subgroups for the maps $\mathcal{N}(T)$.

A map \mathcal{M} with map subgroup M is edge-transitive if and only if E acts transitively on the cosets of $N := N_\Gamma(M)$ in Γ . Since $|E| = 4$, this implies that $|\Gamma : N|$ divides 4, so the different possibilities for N (up to conjugacy) can be found by determining the permutation representations of Γ of degrees 1, 2 and 4 in which E acts transitively. Since $\Gamma = E * \langle R_1 \rangle$, one can consider the actions of E and of R_1 independently.

When $|\Gamma : N| = 1$ we have the trivial action of Γ , in which each R_i induces the identity permutation. Equivalently M is a normal subgroup of Γ , so the corresponding maps \mathcal{M} are the regular maps, those in class 1 in the Graver-Watkins terminology. The basic map \mathcal{N} has a single flag, and it embeds a free edge along part of the boundary of the closed disc $\overline{\mathbb{D}}$.

There are seven subgroups N of Γ of index $|\Gamma : N| = 2$, namely the kernels of the different epimorphisms $\theta : \Gamma \rightarrow C_2$, corresponding to the different choices of which generators R_i to map to the identity (any combination, except all of them). If $R_1, R_2 \in N$ or $R_0, R_1 \in N$ then E is transitive, so these subgroups N are the parent groups for two classes of edge-transitive maps, called 2 and 2^* respectively in [14] (the asterisk indicating that the maps in class 2^* are the duals of those in class 2); however, if $R_0, R_2 \in N$ then E is intransitive, so there is no corresponding class of edge-transitive maps. If just one generator R_i is in N , then E is transitive, and we obtain classes 2^{ex} , 2^P or 2^{ex} as $i = 0, 1$ or 2 (with P indicating Petrie duality). Finally, if no R_i is in N , so that $N = \Gamma^+$, then again E is transitive and we obtain class 2^P ex. The corresponding basic maps \mathcal{N} are all on the closed disc $\overline{\mathbb{D}}$, apart from that for class 2^P ex, which is on the sphere S^2 .

If $|\Gamma : N| = 4$ then E acts transitively if and only if it acts regularly, in which case one can number the four flags of \mathcal{N} so that R_0 and R_2 induce the permutations (12)(34) and (14)(23). If R_1 induces the identity permutation we have class 3. If R_1 induces (14) or (23), giving isomorphic permutation representations of Γ , we have class 4. If R_1 induces (12) or (34) we have class 4^* , and if it induces (13) or (24) we have class 4^P . Finally, if R_1 induces (14)(23), (12)(34) or (13)(24) we have classes 5, 5^* and 5^P respectively. The basic maps \mathcal{N} are on $\overline{\mathbb{D}}$ for types 3, 4 and 4^* , but on the closed Möbius band

for type 4^P . For types 5, 5^* and 5^P the underlying surfaces are S^2 , S^2 and the real projective plane.

Note that if $T = 2^P\text{ex}$, 5 or 5^* then $\mathcal{N}(T)$ is orientable and without boundary, so the same applies to all maps in these classes, since they are coverings of $\mathcal{N}(T)$. When $T = 2^P\text{ex}$ these maps are the orientably regular chiral maps without boundary. The maps in the class 5^P are also without boundary, but they can be orientable or non-orientable.

It is easily seen that 11 of these 14 maps \mathcal{N} are regular, with elementary abelian automorphism and monodromy groups $\text{Aut } \mathcal{N} \cong \Gamma/N$ where N is normal in Γ ; the exceptions are the maps \mathcal{N} for $T = 4$, 4^* and 4^P , which have automorphism groups isomorphic to C_2 and monodromy groups isomorphic to the dihedral group D_4 of order 8.

If T is any of the 14 classes of edge-transitive groups, let us say that a group G is *realised in T* , or is *evenly realised in T* , if T contains a map \mathcal{M} , or an orientable map \mathcal{M} with empty boundary, such that $G \cong \text{Aut } \mathcal{M}$. Let us define the following sets of groups:

$$\mathcal{G}(T) = \{G \mid G \text{ is realised in } T\},$$

$$\mathcal{G}^+(T) = \{G \mid G \text{ is evenly realised in } T\}.$$

The aim of this paper is to determine which members of various families of groups, such as the finite alternating and symmetric groups, are in these sets.

4.3 Operations on edge-transitive maps with boundary

The operations in Ω preserve edge-transitivity. They permute the 14 basic maps \mathcal{N} , and they preserve coverings, so they permute the 14 edge-transitive classes. In particular, the dual pairs are

$$\{2, 2^*\}, \{2\text{ex}, 2^*\text{ex}\}, \{4, 4^*\}, \{5, 5^*\},$$

while the remaining six classes are self-dual. (Of course, this does not mean that the maps themselves are self-dual, only that their duals are in the same class.) Similarly, the Petrie dual pairs are

$$\{2^*, 2^P\}, \{2^*\text{ex}, 2^P\text{ex}\}, \{4^*, 4^P\}, \{5^*, 5^P\},$$

with the remaining six classes are Petrie self-dual. Thus the orbits of Ω are

$$\{1\}, \{2, 2^*, 2^P\}, \{2\text{ex}, 2^*\text{ex}, 2^P\text{ex}\}, \{3\}, \{4, 4^*, 4^P\}, \{5, 5^*, 5^P\},$$

one for each row in Figure 4, where D transposes the first and second terms in each orbit of length 3, while P transposes the second and third. Since Ω preserves automorphism groups of maps, the sets $\mathcal{G}(T)$ are invariant under its action on the classes T . However, the sets $\mathcal{G}^+(T)$ are only $\langle D \rangle$ -invariant.

Duality D preserves possession of a non-empty boundary, and so does Petrie duality P with the following exception: flags fixed by R_0 , corresponding to boundary free edges, are transposed by P with flags fixed by R_0R_2 , corresponding to internal free edges, so that P may cause boundary components to appear or disappear. This exceptional behaviour applies only to the pairs $\{2^*, 2^P\}$ and $\{2^{\text{ex}}, 2^{P\text{ex}}\}$, containing the only classes allowing free edges. In all other cases, Ω preserves the properties of having empty or non-empty boundary

4.4 Ordering of classes

In the Graver-Watkins notation, each class T is denoted by t^σ where t is an integer $1, \dots, 5$ and σ is the symbol $*$, P or \emptyset (denoting the absence of a symbol); in the case $t = 2$ the two orbits of Ω are distinguished by the absence or presence of “ex” after 2^σ . Then $N(T)$ has index

$$n = n(T) = |\Gamma : N(T)| = 1, 2 \text{ or } 4$$

in Γ as $t = 1$, $t = 2$ or $t \geq 3$, so that $\mathcal{N}(T)$ has $n(T)$ flags, corresponding to the orbits of $\text{Aut } \mathcal{M}$ on the flags of \mathcal{M} for each $\mathcal{M} \in T$.

Let us define a relation on the set of 14 classes by writing $T \rightarrow T'$ if each subgroup $N(T)$ is contained in a subgroup $N(T')$, or equivalently there is a covering of maps $\mathcal{N}(T) \rightarrow \mathcal{N}(T')$, which will then have degree equal to the index $|N(T') : N(T)| = n(T)/n(T')$. In this case we will say that T *covers* T' . Clearly this relation is invariant under Ω ; it is reflexive, antisymmetric and transitive, so it is a partial order relation.

It is useful to know which classes T' are covered by each class T , for the following reason. If M is a normal subgroup of $N(T)$ then $N_\Gamma(M) \geq N(T)$, so the corresponding map \mathcal{M} has type T' for some T' covered by T ; if we want to ensure that \mathcal{M} is in class T then we must choose M so that it is not normal in any subgroup $N(T')$ properly containing $N(T)$, and these correspond to the classes $T' \neq T$ covered by T . Equivalently, we must ensure that \mathcal{M} has no automorphisms other than those induced by $N(T)/M$.

Apart from the cases where $T = 4^\sigma$ for some σ and $T' = 1$, all inclusions $N(T) \leq N(T')$ are normal, so that the covering $\mathcal{N}(T) \rightarrow \mathcal{N}(T')$ is regular,

induced by a subgroup of order $n(T)/n(T')$ in $\text{Aut } \mathcal{N}(T)$. Using this, it is easy to determine those classes T' properly covered by each T :

Lemma 4.1 *If T is one of the 14 classes of edge-transitive maps, then the classes T' properly covered by T are as follows:*

1. if $T = 1$ there are none;
2. if $T = 2^\sigma$ or $2^{\sigma\text{ex}}$ for some σ then $T' = 1$;
3. if $T = 3$ then $T' = 1$ or 2^σ for some σ ;
4. if $T = 4^\sigma$ for some σ then $T' = 1$ or 2^σ ;
5. if $T = 5^\sigma$ for some σ then $T' = 1, 2^\sigma$ or $2^{\tau\text{ex}}$, where $\tau \neq \sigma$. \square

We see from this that every proper covering $T \rightarrow T'$ is a composition of (at most two) coverings of degree 2. In order to show that a normal subgroup M of $N(T)$ corresponds to a map \mathcal{M} of type T , and not of some type T' properly covered by T , one may therefore restrict attention to those T' which T covers with degree 2. For each such T' it is then sufficient to choose an arbitrary element $g \in N(T') \setminus N(T)$ and show that $M^g \neq M$, or equivalently that g does not induce an automorphism of \mathcal{M} .

4.5 Parent groups and forbidden automorphisms

Using the Reidemeister-Schreier process, described in [32, §II.4] for example, one can obtain the following presentations for representatives of the orbits of Ω on the 14 parent groups $N(T)$:

Proposition 4.2 *We have the following presentations:*

$$\begin{aligned}
N(1) &= \Gamma = \langle R_0, R_1, R_2 \mid R_i^2 = (R_0 R_2)^2 = 1 \rangle, \\
N(2) &= \langle S_1 = R_1, S_2 = R_1^{R_0}, S_3 = R_2 \mid S_1^2 = S_2^2 = S_3^2 = 1 \rangle, \\
N(2\text{ex}) &= \langle S_1 = R_2, S = R_0 R_1 \mid S_1^2 = 1 \rangle, \\
N(3) &= \langle S_0 = R_1, S_1 = R_1^{R_0}, S_2 = R_1^{R_2}, S_3 = R_1^{R_0 R_2} \mid S_i^2 = 1 \rangle, \\
N(4) &= \langle S_1 = R_1, S_2 = R_1^{R_2}, S = (R_1 R_2)^{R_0} \mid S_1^2 = S_2^2 = 1 \rangle, \\
N(5) &= \langle S = R_1 R_2, S' = S^{R_0} \mid - \rangle.
\end{aligned}$$

In each case, S or S' is an element of infinite order in Γ^+ , while R_i or S_i is a reflection. \square

By applying elements of Ω , that is, by permuting R_0, R_2 and R_0R_2 , one can then obtain presentations for the other parent groups; of course, groups in the same orbit are isomorphic to each other. In particular it is often more convenient to choose

$$N(2^P \text{ex}) = \Gamma^+ = \langle X = R_1R_2, Y = R_0R_2 \mid Y^2 = 1 \rangle \quad (1)$$

as a representative of the third orbit, since it is the parent group of the chiral maps, the subject of a great deal of information in the literature.

One can use these presentations to describe the sets $\mathcal{G}(T)$ of automorphism groups $\text{Aut } \mathcal{M}$ of maps \mathcal{M} in each class T . These all arise as quotients $A = N(T)/M$ of $N(T)$. When $T = 1$, so that $N(T) = \Gamma$, all such quotients are realised as automorphism groups. However, when $T \neq 1$ one has to use Lemma 4.1 to exclude those M with $N_\Gamma(M) = N(T') > N(T)$ for some class T' properly covered by T , corresponding to quotients A with ‘forbidden automorphisms’ induced by elements of $N(T') \setminus N(T)$. As usual, it is sufficient to consider one representative T from each orbit of Ω on classes.

If $T = 1$ (the set of regular maps) then the quotients A of $N(T) = \Gamma$ are those groups generated by elements r_0, r_1 and r_2 of order dividing 2 (the images of R_0, R_1 and R_2), with r_0 and r_2 commuting.

If $T = 2$ then $N(T) \cong C_2 * C_2 * C_2$, and the quotients are the groups A generated by elements s_1, s_2 and s_3 of order dividing 2, with no requirement that two of them should commute. The corresponding map \mathcal{M} is in class $T = 2$ (rather than $T' = 1$) if and only if M is not normalised by the element $R_0 \in \Gamma \setminus N(2)$. Since R_0 , acting by conjugation, transposes S_1 and S_2 and fixes S_3 , this is equivalent to A not having an automorphism transposing s_1 and s_2 and fixing s_3 . A similar restriction applies when $T = 2^*$ or 2^P .

The arguments in the other cases are similar. To summarise the results (see also [47, Condition 3.2]), the 14 classes T are listed in Table 5, with the second column giving the isomorphism type of each parent group $N(T)$, and the third column indicating any forbidden automorphisms; these are given in terms of their effect on generators s_i, s, s' of quotients A , the images of generators S_i, S, S' of cyclic free factors of $N(T)$ defined in Proposition 4.2.

If, instead of $T = 2 \text{ex}$, we use $T = 2^P \text{ex}$ as a representative of the third orbit of Ω , then $N(T) = \Gamma^+$, as in (1), and the forbidden automorphism is that which respectively inverts and fixes the images x and y of the generators X and Y of Γ^+ (or equivalently, since $y^2 = 1$, inverts them both).

We note the following easy consequence of these observations (see [47, Theorem 6.1] for a similar result concerning $\mathcal{G}^+(T)$):

Type T	$N(T)$	forbidden automorphisms
1	$V_4 * C_2$	none
2^σ	$C_2 * C_2 * C_2$	s_1 and s_2 transposed, s_3 fixed
2^σex	$C_2 * C_\infty$	s_1 (or y) fixed, s (or x) inverted
3	$C_2 * C_2 * C_2 * C_2$	double transpositions of generators s_i
4^σ	$C_2 * C_2 * C_\infty$	s_1 and s_2 transposed, s inverted
5^σ	$C_\infty * C_\infty = F_2$	s and s' inverted, transposed or both

Table 5: Parent groups and forbidden automorphisms

Lemma 4.3 *An abelian group A is in $\mathcal{G}(T)$ if and only if either*

- $T = 1$ and $A \cong C_2^r$ with $r = 0, 1, 2$ or 3 , or
- $T = 2^\sigma$ and $A \cong C_2^r$ with $r = 1$ or 2 , or
- $T = 3$ and $A \cong C_2^r$ with $r = 1, 2$ or 3 , or
- $T = 4^\sigma$ and $A \cong C_n$ or $C_2 \times C_n$ where n is even.

Proof. For instance, if $T = 3$ one can realise $A = C_2^r$ for $r = 1, 2$ or 3 by mapping S_0 to the identity element, and S_1, S_2 and S_3 to a generating set of non-identity elements. The rest of the proof is equally straightforward. \square

The following lemma will prove useful later in realising various groups in certain classes, by showing that in many cases it is sufficient to consider just two classes, namely $T = 1$ and 2^Pex .

Lemma 4.4 (a) *If A is a non-abelian group in $\mathcal{G}(1)$, then $A \in \mathcal{G}(T)$ for each class $T = 2^\sigma, 3$ or 4^σ .*

(b) *If A is any group in $\mathcal{G}(2^P \text{ex})$, then $A \in \mathcal{G}(T)$ for each class $T = 2^\sigma \text{ex}, 4^\sigma$ or 5^σ .*

Proof. (a) We have $A = \text{Aut } \mathcal{M}$ for some regular map \mathcal{M} , corresponding to an epimorphism $\Gamma \rightarrow A$, $R_i \mapsto r_i$, so A has generators r_0, r_1 and r_2 satisfying $r_i^2 = (r_0 r_2)^2 = 1$. Since A is non-abelian, replacing \mathcal{M} with its dual if necessary we can assume that $r_1 r_2$ has order $n > 2$.

It is sufficient to consider one representative class T from each orbit of Ω . First let $T = 2$. The presentations of Γ and $N(2)$ show that there is an epimorphism $N(2) \rightarrow \Gamma$ sending S_i to R_{i-1} for $i = 1, 2, 3$. Composing this

with the epimorphism $\Gamma \rightarrow A$, we obtain an epimorphism $\theta : N(2) \rightarrow A$, with $S_i \mapsto s_i := r_{i-1}$ for $i = 1, 2, 3$. An automorphism of A transposing s_1 and s_2 , and fixing s_3 , would send $s_1 s_3 = r_0 r_2$, which has order dividing 2, to $s_2 s_3 = r_1 r_2$, which has order $n > 2$. This is impossible, so $\ker \theta$ is not normal in Γ , and therefore corresponds to an edge-transitive map \mathcal{M}' in class 2 with $\text{Aut } \mathcal{M}' \cong A$.

A similar argument applies to class 4, with $S_i \mapsto r_{i-1}$ as before for $i = 1, 2$, and $S \mapsto s := r_2$. Again no automorphism of A inverting (equivalently, fixing) s transposes s_1 and s_2 , so the corresponding map is in class 4.

Define an epimorphism $N(3) \rightarrow A$, with $S_i \mapsto s_i := r_{i-1}$ for $i = 1, \dots, 3$ as before, and provided $s_1 \neq s_2$ define $S_0 \mapsto s_0 := r_2$. Then $s_0 = s_3$, so the only double transposition of the generators s_i is that transposing s_1 with s_2 and s_0 with s_3 . As before, this cannot extend to an automorphism of A , so the resulting map is in class 3. If $s_1 = s_2$, define $S_0 \mapsto s_0 := r_0$ instead, so $s_0 = s_1 = s_2 \neq s_3$ and there is no double transposition of the generators s_i .

(b) We have $A = \text{Aut } \mathcal{M}$ for some chiral map \mathcal{M} , corresponding to an epimorphism $N(2^{\text{Pex}}) = \Gamma^+ \rightarrow A$, $X \mapsto x, Y \mapsto y$, so that A has generators x and y with $y^2 = 1$. Composing this with the epimorphism $N(5) \rightarrow N(2^{\text{Pex}})$, $S \mapsto X, S' \mapsto Y$ we obtain an epimorphism $\theta : N(5) \rightarrow A$, $S \mapsto x, S' \mapsto y$. Since \mathcal{M} is chiral, x must have order $n > 2$, so an automorphism of A cannot transpose y with $x^{\pm 1}$, and nor can it invert x and y . A similar argument applies to class 4, where we define $\theta : N(4) \rightarrow A$ by $S \mapsto x$ and $S_1, S_2 \mapsto y$. \square

Remarks 1. Lemma 4.3 shows why part (a) of this lemma does not apply to abelian groups.

2. Part (a) does not extend to the remaining classes $T = 2^\sigma \text{ex}$ or 5^σ : for instance, if \mathcal{M} corresponds to a map subgroup M contained in the commutator group Γ' of Γ then A requires three generators and hence cannot be a quotient of a 2-generator group $N(2^\sigma \text{ex})$ or $N(5^\sigma)$. Similarly, part (b) does not extend to the classes $T = 1, 2^\sigma$, or 3 , since Γ^+ has non-trivial quotients of odd order, whereas $N(T)$ does not for these classes T .

Part (c) of the following lemma shows that class 2 can also play a similar role. An element x of a group A is *strongly real* if $x^i = x^{-1}$ for some involution $i \in A$, or equivalently, if x is a product of at most two involutions in A .

Lemma 4.5 (a) *If a group A is generated by involutions a, b and c , where ab, ac and bc do not all have the same order, then $A \in \mathcal{G}(2)$.*

(b) If a group A is generated by elements a, b and c satisfying $abc = 1$, where a is an involution, b is a product of two involutions, and no automorphism of A inverts c , then $A \in \mathcal{G}(2)$.

(c) If $A \in \mathcal{G}(2)$ then $A \in \mathcal{G}(T)$ for each class $T = 2^\sigma, 3$ or 4^σ .

Proof. (a) Without loss of generality we may assume that ac and bc have different orders. There is an epimorphism $N(2) \rightarrow A$ given by $S_i \mapsto s_i := a, b$ or c for $i = 1, 2$ or 3 , and no automorphism of A can transpose s_1 and s_2 while fixing s_3 .

(b) We can write $b = s_1 s_2$ for involutions $s_1, s_2 \in A$, and define $s_3 = a$, giving involutions s_1, s_2, s_3 generating A . If an automorphism transposes s_1 and s_2 while fixing s_3 , it inverts a and b , so composing it with conjugation by a gives an automorphism inverting c . However, no such automorphism exists, so $A \in \mathcal{G}(2)$.

(c) The group $N(4)$ can be obtained from $N(2)$ by taking $S = S_3$ and omitting the relation $S_3^2 = 1$, giving an epimorphism $N(4) \rightarrow N(2)$. Thus any quotient $A = \langle s_1, s_2, s_3 \rangle$ of $N(2)$ is also a quotient of $N(4)$, with $s := s_3$; the forbidden automorphisms are the same in each case, since s has order dividing 2, so $\mathcal{G}(2) \subseteq \mathcal{G}(4)$. Using Ω then allows A to be realised in all classes 2^σ and 4^σ . For class 3 one can extend an epimorphism $N(2) \rightarrow A$ to $N(3) \rightarrow A$ by mapping S_0 to s_3 ; again, this introduces no further forbidden automorphisms. \square

The following observation is easily proved:

Lemma 4.6 *Suppose that A is generated by involutions s_0, s_1, s_2, s_3 , and that for at least two of the three partitions $ij \mid kl$ of $\{0, 1, 2, 3\}$ the products $s_i s_j$ and $s_k s_l$ have different orders. Then $A \in \mathcal{G}(3)$.* \square

Lemma 4.7 *Let $A = \text{Aut } \mathcal{M} = \langle x, y \mid y^2 = 1, \dots \rangle \in \mathcal{G}(2^{\text{Pex}})$ for some chiral map \mathcal{M} , where the image x of X is strongly real. Then $A \in \mathcal{G}(T)$ for each edge-transitive class $T \neq 1$.*

Proof. An involution $a \in A$ inverts x , so $A = \langle s_1, s_2, s_3 \rangle$ where $s_1 = a$, $s_2 = ax$ and $s_3 = y$ all satisfy $s_i^2 = 1$, and hence A is a quotient of $N(2)$. If an automorphism of A transposes s_1 and s_2 and fixes s_3 , it inverts x and fixes y , contradicting the chirality of \mathcal{M} . Thus no such automorphism exists, so $A \in \mathcal{G}(2)$. Then $A \in \mathcal{G}(T)$ for each $T = 2^\alpha, 3$ or 4^α by Lemma 4.5(c), and also for $T = 2^\alpha \text{ex}$ or 5^α by Lemma 4.4(b), so $A \in \mathcal{G}(T)$ for each $T \neq 1$. \square

5 Realising symmetric groups

It is well known that, with a few small exceptions, each finite symmetric or alternating group can be realised as the automorphism group of both a regular map and a chiral map: for instance, these facts can be proved using the methods employed by Conder in [2], where the main emphasis was on realising alternating groups as Hurwitz groups, or more recently by Conder, Hucíková, Nedela and Širáň in [5], where the emphasis was on constructing chiral maps of a given type, rather than with a given automorphism group.

In [47], Širáň, Tucker and Watkins showed that for each integer $n \geq 11$ with $n \equiv 3$ or $11 \pmod{12}$, there are finite, orientable, edge-transitive maps \mathcal{M} in each of the 14 classes T , with $\text{Aut } \mathcal{M}$ isomorphic to the symmetric group S_n . The aim of this section is to extend this result by determining all the pairs n and T such that $S_n \in \mathcal{G}(T)$. (Realisations by orientable maps without boundary will be considered in the next section.) In order to do this, we will need to prove that various sets of permutations generate S_n . The following result is elementary and well-known:

Lemma 5.1 *If t is a transposition (i, j) where i and j are adjacent terms in an n -cycle $c \in S_n$, then c and t generate S_n . \square*

In similar contexts, the following theorem (a simple consequence of [26, Théorème I] and [27, Théorème I], see also [50, Theorem 13.9]) has often been used:

Theorem 5.2 (Jordan) *If G is a primitive subgroup of S_n , containing a cycle of prime length $l \leq n - 3$, then $G \geq A_n$. \square*

(See Theorem 6.1 for a generalisation of Jordan's Theorem, omitting the primality condition.) We will also need the following elementary result:

Lemma 5.3 *If elements x and y generate S_n , where $n \leq 5$, then they are simultaneously conjugate in S_n to their inverses.*

Proof. Suppose that x and y generate S_5 , and are not simultaneously inverted by conjugation. If x is a transposition then in order for $\langle x, y \rangle$ to be transitive, y can have at most two orbits, so it has cycle structure 14 or 23 or 5; in each case a permutation diagram for x and y is symmetric, admitting a reflection which inverts x and y , so x and y are inverted by conjugation in S_5 , against

our assumption. (The four possibilities are shown in Figure 5, where broken and unbroken lines represent the actions of x and y , with cycles of the latter cyclically ordered anticlockwise.) Thus x cannot be a transposition, and hence, by symmetry, neither can y .

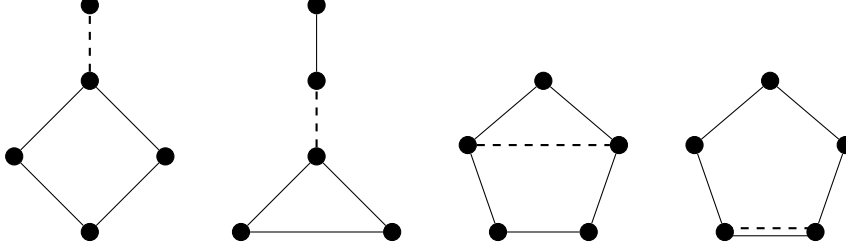


Figure 5: Permutation diagrams for elements of S_5

If x is a double transposition then y must be odd, of cycle structure 14 or 23. Similar arguments using permutation diagrams show that the latter always allows inversion of x and y , while the former allows either inversion of x and y or $G := \langle x, y \rangle \cong \text{AGL}_1(5)$ with $\langle xy^2 \rangle \triangleleft G$. Thus x cannot be a double transposition, and hence neither can y .

If x is a 3-cycle or a 5-cycle then y is odd, with cycle-structure 23 or 14; all the resulting connected diagrams are symmetric, so x and y are inverted. Thus x cannot be a 3-cycle or a 5-cycle, so the same applies to y .

This shows that neither x nor y can be even, so both are odd. But then $z := (xy)^{-1}$ is even, and $\langle x, z \rangle = S_5$, so the preceding arguments show that x and z are inverted, and hence so are x and y .

The arguments for $n \leq 4$ are similar, and even more elementary, so they are omitted. \square

The following result proves the statements in Theorem 1.1 concerning symmetric groups:

Theorem 5.4 *The symmetric group S_n is the automorphism group of a map in an edge-transitive class T if and only if one of the following holds:*

- $T = 1$ and $n \geq 1$;
- $T = 2^\sigma$, 3 or 4^σ for some σ and $n \geq 2$;
- $T = 2^\sigma \text{ ex}$ or 5^σ for some σ and $n \geq 6$.

Proof. Lemma 4.3 deals with the cases $n \leq 2$, so we may assume that $n \geq 3$. We can map the standard generators R_i of $N(1) = \Gamma$ to permutations $r_i \in S_n$, where

$$r_1 = (1)(2, n)(3, n-1) \dots, \quad r_2 = (1, 2)(3, n)(4, n-1) \dots$$

are involutions in S_n with

$$r_1 r_2 = (1, 2, \dots, n),$$

and where

$$r_0 = (1, 2)$$

is an involution commuting with r_2 . This defines a homomorphism $\Gamma \rightarrow S_n$, and since $\langle r_1 r_2, r_0 \rangle = S_n$ by Lemma 5.1 it is an epimorphism. The kernel M is then the map subgroup of a regular map \mathcal{M} with $\text{Aut } \mathcal{M} \cong S_n$, so $S_n \in \mathcal{G}(1)$. Since S_n is non-abelian, it follows from Lemma 4.4(a) that there are also edge-transitive maps in classes $T = 2^\sigma$, 3 and 4^σ with automorphism group S_n for each σ and each $n \geq 3$.

Now suppose that $T = 2^P \text{ex}$, so that $N(T) = \Gamma^+ = \langle X, Y \mid Y^2 = 1 \rangle$. If $n \geq 6$ let us map the generators X and Y to the permutations

$$x = (1, 2, \dots, n-1) \quad \text{and} \quad y = (1, 3)(2, 4)(n-1, n)$$

in S_n . The group $G = \langle x, y \rangle$ is clearly transitive, and in fact doubly transitive since it contains x , so it is primitive. If $n \geq 7$ then

$$[x, y] = (1, n-1, n, 3, 5),$$

a cycle of length 5, so Jordan's Theorem implies that $G \geq A_n$ provided $n \geq 8$, and since y is an odd permutation it follows that $G = S_n$. In fact this also holds for $n = 7$ since the only primitive groups of this degree containing a 5-cycle are A_7 and S_7 . In each case it is easy to check, for instance by drawing a permutation diagram, that no automorphism of S_n (necessarily inner) inverts both x and y . This argument fails for $n = 6$, but in this case we can map X and Y to

$$x = (1, \dots, 6) \quad \text{and} \quad y = (1, 2)(3, 5).$$

Then $xy = (2, 5, 6)(3, 4)$ so $(xy)^3 = (3, 4)$ and hence $\langle x, y \rangle = S_6$; again a permutation diagram shows that no inner automorphism inverts x and y ,

and since the outer automorphisms of S_6 send 6-cycles to products of disjoint 2- and 3-cycles, they are also eliminated. By Lemma 4.4(b) the result follows for the remaining classes 2^σex and 5^σ for all $n \geq 6$.

If $n \leq 5$ then Lemma 5.3 shows that there are no edge-transitive maps in classes 2^σex or 5^σ with automorphism group S_n : in each case the two generators of a quotient group are inverted by some automorphism of S_n . \square

Cayley's Theorem gives the following consequence of Theorem 5.4:

Corollary 5.5 *If F is any finite group then for each edge-transitive class T there is a compact map \mathcal{M} in T such that F is isomorphic to a subgroup of $\text{Aut } \mathcal{M}$.* \square

In fact, by using an embedding theorem of Schupp [45], and allowing non-compact maps, this result can be extended from finite groups to all countable groups, as shown in Section 11.2.

Note that in the cases where $T = 2^\sigma$, 3 or 4^σ and $n = 2$, the maps used in the proof of Theorem 5.4 have non-empty boundaries. This cannot be avoided, since in order to exclude forbidden automorphisms some reflection $S_i \in N(T)$ must be mapped to the identity. In the next section we will consider realisations of symmetric groups as the automorphism groups of edge-transitive maps which are orientable and without boundary.

6 Evenly realising symmetric groups

We saw in Theorem 5.4 that for almost all pairs T and n there is an edge-transitive map in class T with automorphism group S_n . In many cases the maps constructed there are non-orientable, and in a few cases they have non-empty boundaries. We will now determine, for each edge-transitive class T , the values of n for which S_n is the automorphism group of an orientable map without boundary in the class T , that is, $S_n \in \mathcal{G}^+(T)$.

For brevity, in this case we will often simply say that S_n is *evenly realised* in class T , since a map \mathcal{M} is orientable and without boundary if and only if the corresponding map subgroups M are contained in the even subgroup Γ^+ of index 2 in Γ . If $T = 2^P \text{ex}$, 5 or 5^* then $N(T) \leq \Gamma^+$ (since $\mathcal{N}(T)$ is a map on the sphere), so in such cases *all* maps in T are orientable and without boundary, and Theorem 5.4 gives the required result. For the remaining classes T , we need to consider whether one can find a quotient $N(T)/M$

of $N(T)$ isomorphic to S_n , with no forbidden automorphisms, and with the additional requirement that $M \leq \Gamma^+$. In terms of the induced epimorphism $N(T) \rightarrow S_n$, this last condition is equivalent to requiring that all orientation-reversing generators of $N(T)$ should be sent to odd permutations.

As before, we will need to show that various sets of permutations generate S_n . In addition to Lemma 5.1 and Theorem 5.2 (Jordan's Theorem) we will need the following result, which uses the classification of finite simple groups to remove the primality condition in Jordan's Theorem (see [21]):

Theorem 6.1 *If G is a primitive subgroup of S_n , containing a cycle of length $l \leq n - 3$, then $G \geq A_n$.* \square

We will also use the following simple corollary (see [22, Lemma 6.4(i)]):

Corollary 6.2 *If G is a transitive subgroup of S_n , containing a permutation consisting of two cycles with mutually coprime lengths $l, m > 1$, and no fixed points, then $G \geq A_n$.* \square

The main result of this section is the following:

Theorem 6.3 *The symmetric group S_n is the automorphism group of an orientable map without boundary in an edge-transitive class T if and only if one of the following holds:*

- (a) $T = 1$ and $n \neq 1, 5$ or 6 ;
- (b) $T = 2$ or 2^* and $n \neq 1, 2, 5$ or 6 , or $T = 2^P$ and $n \geq 3$;
- (c) $T = 2^{\text{ex}}$ or $2^{*\text{ex}}$ and $n \geq 7$, or $T = 2^P$ and $n \geq 6$;
- (d) $T = 3$ and $n \geq 3$;
- (e) $T = 4^\sigma$ for some σ and $n \geq 3$;
- (f) $T = 5^\sigma$ for some σ and $n \geq 6$.

The six parts of this theorem correspond to the six orbits of Ω on the classes T , and thus on the isomorphism classes of parent groups $N(T)$. However, in contrast with Theorem 5.4, the results in (b) and (c) vary for different classes in the same orbit, since we now have to take into account which generators of $N(T)$ preserve or reverse orientation.

6.1 Proof of (a), with $T = 1$

First let $T = 1$, the class of regular maps, so that $N(T) = \Gamma$ and hence $N(T) \cap \Gamma^+ = \Gamma^+$. Given any epimorphism $\theta : \Gamma \rightarrow S_n, R_i \mapsto r_i$, the kernel M is contained in a unique subgroup $\theta^{-1}(A_n)$ of index 2 in Γ , and this is Γ^+ if and only if it does not contain any R_i , that is, r_i is an odd permutation for each $i = 0, 1, 2$. We will therefore look for triples of odd involutions r_i generating S_n , with r_0 and r_2 commuting.

Case 1 Let $n \equiv 3 \pmod{4}$, say $n = 4k + 3 \geq 3$. Define

$$\begin{aligned} r_0 &= (1)(2, n)(3, n-1) \dots (2k+2, 2k+3), \\ r_1 &= (1, 2)(3, n)(4, n-1) \dots (2k+2, 2k+4)(2k+3), \\ r_2 &= (2k+2, 2k+3), \end{aligned}$$

so that each r_i is an odd involution, and r_2 commutes with r_0 . Then

$$r_0 r_1 = (1, 2, \dots, n), \quad \text{so} \quad \langle r_0 r_1, r_2 \rangle = S_n$$

by Lemma 5.1, and we have an epimorphism $\Gamma \rightarrow S_n$ with the required properties.

Case 2 Let $n \equiv 0 \pmod{4}$, say $n = 4k \geq 4$. Define

$$\begin{aligned} r_0 &= (1)(2, n-1)(3, n-2) \dots (2k, 2k+1)(n), \\ r_1 &= (1, 2)(3, n-1)(4, n-2) \dots (2k, 2k+2)(2k+1)(n), \\ r_2 &= (1, n). \end{aligned}$$

Again, each r_i is an odd involution, and r_2 commutes with r_0 . Now

$$r_0 r_1 = (1, 2, \dots, n-1)(n),$$

so

$$r_0 r_1 r_2 = (1, 2, \dots, n) \quad \text{and} \quad \langle r_0 r_1 r_2, r_2 \rangle = S_n,$$

giving the required epimorphism $\Gamma \rightarrow S_n$.

These two cases rely on r_0 and r_1 being natural generators of a ‘large’ dihedral subgroup of S_n . The remaining cases are less straightforward, since the corresponding dihedral subgroup is not large enough for our purposes.

Case 3 Let $n \equiv 1 \pmod{4}$, say $n = 4k + 1$, and suppose that $n \geq 9$. Define

$$r_0 = (1, 2)(3, 4) \dots (n-4, n-3) \quad \text{and}$$

$$r_2 = (3, 5)(4, 6)(7, 9) \dots (n-5, n-3)(n-2, n-1),$$

commuting odd involutions, and define r_1 to be the odd involution

$$(1, 3)(5, 7)(6, 8)(9, 11)(10, 12) \dots (n-8, n-6)(n-7, n-5)(n-4, n-2)(n-1, n)$$

or

$$(1, 3)(5, 8)(6, 7)(9, 11)(10, 12) \dots (n-8, n-6)(n-7, n-5)(n-4, n-2)(n-1, n)$$

as k is even or odd.

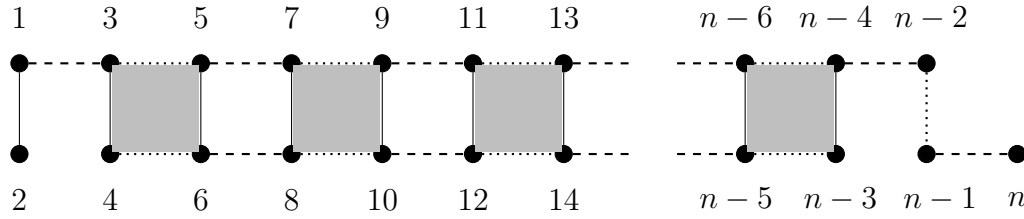


Figure 6: The permutations r_i for $n = 4k + 1$, k even

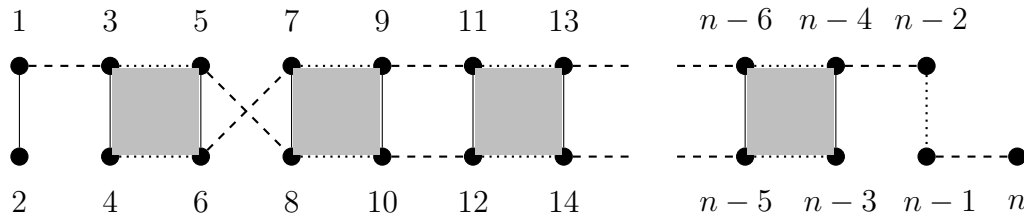


Figure 7: The permutations r_i for $n = 4k + 1$, k odd

These permutations are shown in Figures 6 and 7 respectively, with unbroken, dashed and dotted lines representing 2-cycles of r_0 , r_1 and r_2 . For clarity, the squares representing regular orbits of the subgroup $\langle r_0, r_2 \rangle \cong V_4$ are coloured grey.

The permutations r_i generate a transitive group $G \leq S_n$. Now $r_0 r_1 r_2$ consists of two cycles

$$(1, 2, 5, 10, 13, 18, \dots, n-7, n-4, n-5, n-10, n-13, \dots, 7, 4) \quad \text{and}$$

$$(3, 6, 9, 14, 17, \dots, n-3, n-1, n, n-2, n-6, 11, 8),$$

or

$$(1, 2, 5, 9, 14, 17, \dots, n-4, n-5, n-10, \dots, 11, 8, 4) \quad \text{and}$$

$$(3, 6, 10, 13, 18, \dots, n-3, n-1, n, n-2, n-6, n-9, \dots, 12, 7),$$

as k is even or odd. (When $n = 9$ or $n = 13$ the cycles of $r_0 r_1 r_2$ reduce to

$$(1, 2, 5, 4)(3, 6, 8, 9, 7) \quad \text{or} \quad (1, 2, 5, 9, 8, 4)(3, 6, 10, 12, 13, 11, 7)$$

respectively.) In each case, the two cycles of $r_0 r_1 r_2$ have mutually coprime lengths $2k$ and $2k+1$, so Corollary 6.2 implies that $G \geq A_n$. Since G contains odd permutations we have $G = S_n$ as required.

This construction fails if $n = 5$. Indeed, the odd involutions in S_5 are the transpositions, and it is easy to see that three of these cannot generate a transitive group, so S_5 cannot be the automorphism group of an orientable regular map. However, by Theorem 5.4 there are regular maps with automorphism group S_5 : the only such maps are the non-orientable map of genus 5 and type $\{4, 5\}_6$, and its dual, listed as N5.1 in Conder's list of non-orientable regular maps [3].

Case 4 Let $n \equiv 2 \pmod{4}$, say $n = 4k + 2$. First suppose that $n \geq 18$. Define

$$r_0 = (1, 2)(3, 4) \dots (n-1, n),$$

$$r_1 = (2, 4)(6, 8)(9, 11)(10, 12)(13, 15)(14, 16) \dots$$

$$\dots (n-13, n-11)(n-12, n-10)(n-9, n-7)(n-5, n-3)(n-2, n),$$

$$r_2 = (3, 5)(4, 6)(7, 9)(8, 10) \dots (n-7, n-5)(n-6, n-4)(n-3, n-2).$$

These permutations, shown in Figure 8, are odd involutions in S_n , and r_0 commutes with r_2 . The element $r_1 r_2$ fixes 1 and $n-1$, has two 2-cycles $(3, 5)$ and $(n-6, n-4)$, and has two cycles

$$(2, 6, 10, \dots, n-8, n-10, n-14, \dots, 8, 4)$$

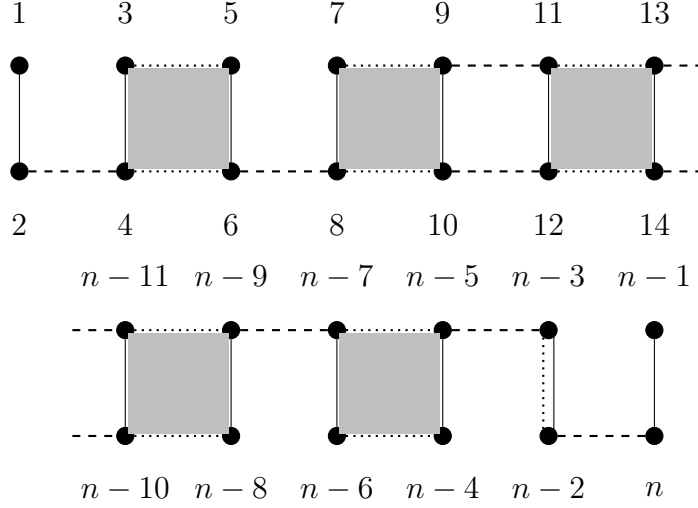


Figure 8: The permutations r_i for $n = 4k + 2$

and

$$(7, 9, 13, 17, \dots, n-5, n-2, n, n-3, n-7, \dots, 11)$$

of lengths $2k-3$ and $2k-1$.

We need to show that the group $G = \langle r_0, r_1, r_2 \rangle$ is primitive, so suppose not. Now $2(2k-3)$ is coprime to $2k-1$, so $(r_1 r_2)^{2(2k-3)}$ is a cycle c of length $2k-1$. This is coprime to n , so c cannot be a union of blocks of imprimitivity, and hence some block B meets the support C of c and its complement. Since B contains a fixed point of c it is invariant under c , and hence contains all of C . Thus $|B| > 2k-1$, and since $|B|$ divides n it follows that $|B| = 2k+1$ and there are just two blocks. Since r_1 and r_2 have fixed points they preserve the blocks, so r_0 must transpose them. This is impossible, since r_0 and r_2 both transpose $n-3$ and $n-2$. Thus G is primitive, so we can apply Theorem 6.1 to the cycle c to show that $G \geq A_n$. Since G contains odd permutations we have $G = S_n$, as required.

Now suppose that $n = 14$. In this case let

$$r_0 = (1, 2)(3, 4) \dots (13, 14)$$

as before, but let

$$r_1 = (2, 4)(5, 7)(6, 8)(9, 11)(12, 14),$$

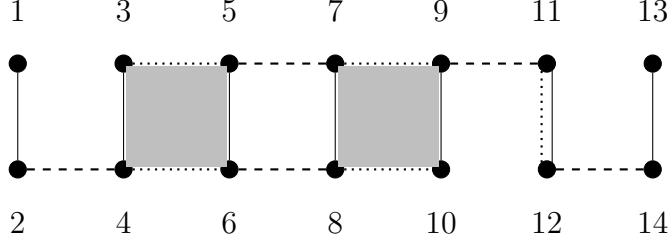


Figure 9: The permutations r_i for $n = 14$

$$r_2 = (3, 5)(4, 6)(7, 9)(8, 10)(11, 12),$$

as shown in Figure 9. Then

$$r_1 r_2 = (2, 6, 10, 8, 4)(3, 5, 9, 12, 14, 11, 7),$$

with cycle-structure $1^2, 5, 7$, so that $c := (r_1 r_2)^5$ is a 7-cycle. Suppose that $G = \langle r_0, r_1, r_2 \rangle$ is imprimitive. Since c has fixed points, the only possible blocks are the support of c and its complement. However, r_0 neither preserves nor transposes these two sets, so G is primitive. Now Jordan's Theorem 5.2, applied to c , implies that $G = S_{14}$, as required.

When $n = 10$ let

$$r_0 = (1, 2)(3, 4) \dots (9, 10)$$

as before, but let

$$r_1 = (2, 4)(5, 7)(8, 10),$$

$$r_2 = (3, 5)(4, 6)(7, 8),$$

as shown in Fig. 10. In this case

$$r_1 r_2 = (2, 6, 4)(3, 5, 8, 10, 7),$$

so $(r_1 r_2)^3$ is a 5-cycle, and an argument similar to that used for $n = 14$ shows that $\langle r_0, r_1, r_2 \rangle = S_{10}$.

However, we will now show that S_6 cannot be the automorphism group of an orientable regular map. The generators r_i would need to be either transpositions or triple transpositions. Now S_6 has an outer automorphism transposing these two conjugacy classes. Clearly S_6 cannot be generated

By the proof of part (a), if $n \neq 1, 5$ or 6 then S_n is evenly realised for class 1 by means of an epimorphism $\Gamma \rightarrow S_n$, $R_i \mapsto r_i$ ($i = 0, 1, 2$), so we can define $s_i = r_{i-1}$, giving three odd involutions generating S_n . If $n \geq 3$ then s_1 commutes with s_3 , whereas s_2 does not (since r_2 is not central in S_n), so there is no automorphism fixing s_2 and transposing s_1 and s_3 . This evenly realises S_n for each of the three classes $T = 2^\sigma$ for all $n \neq 1, 2, 5$ or 6 .

Clearly the symmetric groups S_1 and S_2 cannot be evenly realised, and neither can S_5 for $T = 2$ or 2^* since it cannot be generated by three transpositions. However, S_5 can be evenly realised for $T = 2^P$ by taking

$$s_1 = (1, 2), \quad s_2 = (3, 4) \quad \text{and} \quad s_3 = (1, 3)(4, 5).$$

The group G which these generate is transitive and hence primitive, and it contains a transposition s_1 , so $G = S_5$. The centraliser of s_3 in $\text{Aut } G = S_5$ is the Klein four-group fixing 2, and this cannot transpose s_1 and s_2 .

To evenly realise S_6 for $T = 2^P$ take

$$s_1 = (1, 3), \quad s_2 = (1, 5)(2, 3)(4, 6) \quad \text{and} \quad s_3 = (1, 2)(3, 4).$$

These generate a transitive group G , and since $s_1 s_2 s_3 = (2, 4, 6, 3, 5)$ this group is doubly transitive and hence primitive. Since $s_1 \in G$ we have $G = S_6$. Since $s_1 s_3 = (1, 4, 3, 2)$ has order 4, while $s_2 s_3 = (1, 5, 2, 4, 6, 3)$ has order 6, no automorphism can fix s_3 while transposing s_1 and s_2 .

However, S_6 cannot be evenly realised for $T = 2$ or 2^* . It clearly cannot be generated by three transpositions, and hence (by applying the outer automorphism of S_6) the same applies to three triple transpositions. The only possibilities are therefore two transpositions and one triple transposition, or vice versa, and again these are equivalent under the outer automorphism. In the former case we can assume without loss of generality that s_1, s_2 and s_3 are $(1, 2)(3, 4)(5, 6)$, $(2, 4)$ and $(3, 5)$, in which case they generate an imprimitive group, preserving the relation of congruence mod (2). Thus S_6 cannot be evenly realised for these two classes.

6.3 Proof of (e), with $T = 4^\sigma$

If $T = 4^\sigma$ we have

$$N(T) = \langle S_1, S_2, S \mid S_i^2 = 1 \rangle \cong C_2 * C_2 * C_\infty,$$

and for epimorphisms $N(T) \rightarrow G = \langle s_1, s_2, s \rangle$ there is one forbidden automorphism, transposing s_1 and s_2 , while inverting s . If $T = 4$ or 4^* then each S_i is orientation-reversing while S is orientation-preserving, but if $T = 4^P$ then all three generators of $N(T)$ are orientation-reversing.

To evenly realise S_n for $n \neq 1, 2, 5$ or 6 we can use the generators s_1, s_2 and s used for types $T = 2^\sigma$.

To evenly realise S_5 for $T = 4^\sigma$, take

$$s_1 = (1, 2), \quad s_2 = (3, 4) \quad \text{and} \quad s = (2, 3, 4, 5).$$

These generate a group which is transitive and hence primitive since its degree is prime; since it contains s_1 this group is S_5 . Since $s_1 s = (1, 3, 5, 5, 2)$ and $s_2 s^{-1} = (2, 5, 4)$ have different orders, there is no automorphism transposing s_1 and s_2 and inverting s .

To evenly realise S_6 for $T = 4^\sigma$, take

$$s_1 = (1, 2), \quad s_2 = (3, 4) \quad \text{and} \quad s = (1, 5, 6)(2, 3).$$

These generate a transitive group G , and since it contains $s_1 s = (1, 3, 2, 5, 6)$ it is doubly transitive and hence primitive. Since $s_1 \in G$ we have $G = S_6$. Since $s_2 s^{-1} = (1, 6, 5)(2, 3, 4)$ has order 3, no automorphism can transpose s_1 and s_2 while inverting s .

6.4 Proof of (d), with $T = 3$

Now let $T = 3$, the class of just-edge-transitive maps. We have

$$N(3) = \langle S_0, \dots, S_3 \mid S_i^2 = 1 \rangle \cong C_2 * C_2 * C_2 * C_2,$$

where each S_i is a reflection, so we require epimorphisms $N(3) \rightarrow S_n$, $S_i \mapsto s_i$, with each s_i an odd involution. The forbidden automorphisms are those inducing double transpositions of these generators.

Write $n = m + r$ where $m \equiv 3 \pmod{4}$ and $r = 0, 1, 2$ or 3 . Let

$$s_0 = (1, 2)(3, 4) \dots (m-2, m-1),$$

$$s_1 = (2, 3)(4, 5) \dots (m-1, m),$$

odd involutions such that

$$s_0 s_1 = (1, 3, 5, \dots, m, m-1, m-3, \dots, 2)$$

is an m -cycle, fixing r points. Let $G := \langle s_0, s_1, s_2, s_3 \rangle$, where s_2 and s_3 are odd involutions to be defined in the various cases below.

If $n \equiv 3 \pmod{4}$, so that $m = n$ and $r = 0$, define

$$s_2 = s_3 = (1, 2).$$

Then $\langle s_0 s_1, s_2 \rangle = S_n$ by Lemma 5.1, so we have an epimorphism $N(3) \rightarrow G = S_n$ with kernel $M \leq \Gamma^+$. Since s_0, s_2 and s_3 commute with each other, whereas none of them commutes with s_1 , no automorphism of S_n can induce a double transposition on these four generators, so the corresponding map is in class 3.

Similar arguments apply in the other cases. If $n \equiv 0 \pmod{4}$, so that $m = n - 1$ and $r = 1$, define

$$s_2 = s_3 = (1, n),$$

so that

$$s_0 s_1 s_2 = (1, 3, 5, \dots, n-1, n-2, n-4, \dots, 2, n)$$

is an n -cycle and $\langle s_0 s_1 s_2, s_2 \rangle = S_n$.

If $1 < n \equiv 1 \pmod{4}$, so that $m = n - 2$ and $r = 2$, define

$$s_2 = (1, n) \quad \text{and} \quad s_3 = (2, n-1),$$

so that

$$s_0 s_1 s_2 = (1, 3, 5, \dots, n-2, n-3, n-5, \dots, 4, n-1, 2, n)$$

is an n -cycle and $\langle s_0 s_1 s_2 s_3, s_2 \rangle = S_n$.

If $2 < n \equiv 2 \pmod{4}$, so that $m = n - 3$ and $r = 3$, define

$$s_2 = (1, n)(2, n-1)(3, n-2) \quad \text{and} \quad s_3 = (1, n),$$

so that

$$s_0 s_1 s_2 = (1, n-2, 3, 5, \dots, n-3, n-4, n-6, \dots, 4, n-1, 2, n)$$

is an n -cycle and $\langle s_0 s_1 s_2, s_3 \rangle = S_n$. In all three cases, it is easy to check that the cycle-structures of the generators and their commuting relations prevent any forbidden automorphisms from arising.

However, if $n = 2$ the only choice for the generators is $s_i = (1, 2)$ for all i , so there are forbidden automorphisms, while if $n = 1$ there are no odd involutions to choose.

Example When $n = 3$ the construction described above leads to a just-edge-transitive map on the sphere, in which a vertex of valency 6 is joined by double edges to three vertices of valency 2; the faces are three digons and one hexagon.

6.5 Proof of (c), with $T = 2^\sigma \text{ex}$

If $T = 2^\sigma \text{ex}$ we have

$$N(T) = \langle S_1, S \mid S_1^2 = 1 \rangle \cong C_2 * C_\infty,$$

with S preserving orientation, but S_1 preserving it if and only if $\sigma = P$ (so that T is the class of orientably regular chiral maps). The forbidden automorphism is that which fixes s_1 and inverts s .

The proof of Theorem 5.4 gives a generating pair

$$s_1 = (1, 3)(2, 4)(n-1, n) \quad \text{and} \quad s = (1, 2, \dots, n-1)$$

for S_n for each $n \geq 7$, with s_1 an odd involution and with no forbidden automorphism, so this evenly realises such groups S_n for each class 2^σex . When $n = 6$ the generators

$$(1, \dots, 6) \quad \text{and} \quad (1, 2)(3, 5)$$

used there deal with the class 2^Pex , but for classes 2ex and 2^*ex we need the involution s_1 to be odd, that is, either a transposition or a triple transposition. Suppose first that s_1 is a transposition; if $\langle s_1, s \rangle$ is to be transitive then s can have at most two cycles, in which case one easily sees by drawing permutation diagrams that there is always a forbidden automorphism. Thus s_1 cannot be a transposition, and by applying the outer automorphism of S_6 we can eliminate the case where s_1 is a triple transposition.

By the proof of Theorem 5.4, S_n cannot be evenly realised by any class 2^σex if $n \leq 5$.

6.6 Proof of (f), with $T = 5^\sigma$

If $T = 5^\sigma$ then

$$N(T) = \langle S, S' \mid -- \rangle \cong F_2,$$

with S and S' both even if $\sigma = \emptyset$ or $*$, but both odd if $\sigma = P$. The forbidden automorphisms are those inverting, transposing, or inverting and transposing their images s and s' .

First suppose that $n \geq 7$. If $T = 5$ or 5^* , or if $T = 5^P$ and n is odd, then the generators

$$(1, 3)(2, 4)(n-1, n) \quad \text{and} \quad (1, 2, \dots, n-1)$$

used above for the classes 2^σex can be used as s and s' . We saw earlier that they are not simultaneously inverted, and since they have different orders, no automorphism of S_n can transpose them or invert and transpose them. If $T = 5^P$ and n is even let

$$s = (1, 2)(3, 4, 5) \quad \text{and} \quad s' = (1, 2, \dots, n).$$

These generate S_n since $s^3 = (1, 2)$. They have different orders, and a permutation diagram shows that they are not simultaneously inverted by conjugation, so there are no forbidden automorphisms.

If $n = 6$ then for any $T = 5^\sigma$ let

$$s = (1, 2, 5, 3) \quad \text{and} \quad s' = (1, 2, \dots, 6).$$

Then

$$s^2 s' = (1, 6)(2, 4, 5),$$

so the elements $(s^2 s')^3 = (1, 6)$ and s' generate S_6 . They have different orders, so no automorphism can transpose them or invert and transpose them. A permutation diagram shows that no inner automorphism can invert them, and outer automorphisms cannot do so since they send 6-cycles to elements with cycle structure 123.

Finally, Lemma 5.3 shows that $S_n \notin \mathcal{G}(5^\sigma)$ if $n \leq 5$. □

7 Realising alternating groups

In order to prove an analogue of Theorem 5.4 for the alternating groups, we need to have some suitable generators for these groups.

Lemma 7.1 *The following sets of elements each generate the alternating group A_n :*

- (a) $(1, 2, 3), (2, 3, 4), \dots, (n-2, n-1, n)$ for each $n \geq 3$;
- (b) $(1, 2, 3), (1, 3, 4), \dots, (1, n-1, n)$ for each $n \geq 3$;
- (c) $a := (k, k+1, k+2), c := (1, 2, \dots, n)$ for each odd $n \geq 3$ and any k ;
- (d) $a := (1, k, k+1), c := (2, 3, \dots, n)$ for each even $n \geq 4$ and $k \neq 1, n$;

(Here we reduce the entries of $a \bmod (n)$ if they lie outside $\{1, 2, \dots, n\}$.)

Proof. Statements (a) and (b) can easily be proved by induction on n , using the fact that A_n is a maximal subgroup of A_{n+1} . For (c) and (d), conjugating a by powers of c gives the generators in (a) or (b). \square

The following result proves the statements in Theorem 1.1 concerning the alternating groups:

Theorem 7.2 *The alternating group A_n is the automorphism group of a map in an edge-transitive class T if and only if one of the following holds:*

- $T = 1$, and $n = 1, 2, 5$ or $n \geq 9$;
- $T = 2^\sigma$ for some σ or $T = 3$, and $n \geq 5$;
- $T = 2^\sigma \text{ex}$ for some σ , and $n \geq 8$;
- $T = 4^\sigma$ for some σ , and $n \geq 4$;
- $T = 5^\sigma$ for some σ , and $n \geq 7$.

Proof. The cases $n \leq 3$ are covered by Lemma 4.3, and A_4 is easily dealt with, so we may assume that $n \geq 5$. If $n \neq 6$ then all automorphisms of A_n are induced by conjugation in S_n , which preserves cycle structure, so for convenience we will first consider larger values of n , leaving smaller values until later.

First let $T = 1$, so that $N(T) = \Gamma \cong V_4 * C_2$. Nuzhin [40], responding to a question of Mazurov (see Section 9), showed that A_n is a quotient of Γ if and only if $n = 5$ or $n \geq 9$. Specifically, he showed that in these cases one can map the standard generators R_i of Γ to generators r_i of A_n , where

$$r_0 = (1, 4)(2, 3)(5, 6)(n-2, n-1) \quad \text{if } n = 4k+3 \geq 11,$$

$$r_0 = (1, 2)(3, 4) \quad \text{otherwise,}$$

and r_2 and r_1 are respectively

$$(1, 4)(2, 3), (2, 3)(4, 5) \quad \text{if } n = 5,$$

$$(1, 2)(3, 4) \dots (n-2, n-1), (2, 3)(4, 5) \dots (n-1, n) \text{ if } n = 4k+1 \geq 9,$$

$$(3, 4)(5, 6) \dots (n-1, n), (2, 3)(4, 5) \dots (n-2, n-1) \text{ if } n = 4k+2 \geq 10,$$

$$(1, 2)(3, 4) \dots (n-4, n-3), (4, 5)(6, 7) \dots (n-1, n) \text{ if } n = 4k+3 \geq 11,$$

$$(1, 2)(3, 4) \dots (n-1, n), (2, 3)(4, 5)(6, 7) \dots (n-2, n-1)(1, n) \text{ if } n = 4k \geq 12.$$

This deals with $T = 1$, and hence by Lemma 4.4(a) it also realises A_n for $n = 5$ and $n \geq 9$, when $T = 2^\sigma$, 3 or 4^σ . However, for these classes we will later give direct arguments which realise A_n for a wider range of values of n .

We next consider $T = 2^P \text{ex}$, with $N(T) = \Gamma^+ = \langle X, Y \mid Y^2 = 1 \rangle$. For even $n \geq 8$ let

$$x = (2, 3, \dots, n) \quad \text{and} \quad y = (1, 2)(3, 4),$$

so $G := \langle x, y \rangle$ is 2-transitive and hence primitive. Now

$$[y, x] = y \cdot y^x = (1, 2)(3, 4) \cdot (1, 3)(4, 5) = (1, 2, 3, 5, 4),$$

so by Jordan's Theorem $G = A_n$. We can therefore define an epimorphism $\Gamma^+ \rightarrow A_n$ by $X \mapsto x$ and $Y \mapsto y$, so A_n is a quotient of Γ^+ . Since $n \neq 6$ every automorphism of A_n is induced by conjugation in S_n ; by inspection, no permutation inverts x and centralises y , so there are no forbidden automorphisms. For odd $n \geq 9$, let

$$x = (1, 2, \dots, n) \quad \text{and} \quad y = (1, 2)(3, 6),$$

and suppose that $G := \langle x, y \rangle$ is imprimitive. The only non-trivial equivalence relations invariant under $\langle x \rangle$ are those of congruence mod (d) for some d dividing n , with $3 \leq d \leq n/3$ since n is odd. Each equivalence class contains at least three elements, and y must transpose the classes $[1]$ and $[2]$, which is impossible since it moves only four elements. Thus G is primitive. Now

$$[y, x^2] = (1, 2)(3, 6, 4)(5, 8),$$

so $[y, x^2]^2$ is a 3-cycle and hence $G = A_n$ by Jordan's Theorem. As before, this shows that A_n is a quotient of Γ^+ , and since no permutation inverts x and

commutes with y , there are no forbidden automorphisms. By Lemma 4.4(b) this realises A_n for $n \geq 8$, when $T = 2^\sigma \text{ex}$ or 5^σ .

We now consider small values of n not dealt with above. In the case $T = 1$, Nuzhin [40] has already shown that there are regular maps with automorphism group isomorphic to A_n for $n = 5$ but not for $n = 6, 7$ or 8 . (For example, a simple argument using permutation diagrams shows that if a Klein four group and an involution generate a transitive subgroup $G \leq A_6$, then either G is imprimitive, with three blocks of size 2, or $G \cong L_2(5)$; the latter possibility realises A_5 , which is isomorphic to $L_2(5)$.) This deals with $T = 1$, and by Lemma 4.4(a) it also realises A_5 in the classes 2^σ , 3 and 4^σ .

If $T = 2^\sigma$ then $N(T) \cong C_2 * C_2 * C_2$. For $n = 6$ take

$$s_1 = (1, 2)(3, 4), \quad s_2 = (2, 6)(4, 5) \quad \text{and} \quad s_3 = (2, 3)(4, 5),$$

so that

$$s_1 s_2 = (1, 6, 2)(3, 5, 4), \quad s_1 s_3 = (1, 3, 5, 4, 2) \quad \text{and} \quad s_2 s_3 = (2, 6, 3).$$

For $n = 7$ take

$$s_1 = (1, 2)(3, 4), \quad s_2 = (2, 6)(5, 7) \quad \text{and} \quad s_3 = (2, 3)(4, 5),$$

so that

$$s_1 s_2 = (1, 6, 2)(3, 4)(5, 7), \quad s_1 s_3 = (1, 3, 5, 4, 2) \quad \text{and} \quad s_2 s_3 = (2, 6, 3)(4, 5, 7).$$

For $n = 8$ take

$$s_1 = (1, 2)(3, 4)(5, 6)(7, 8), \quad s_2 = (1, 3)(4, 6) \quad \text{and} \quad s_3 = (3, 4)(6, 7),$$

so that

$$s_1 s_2 = (1, 2, 3, 6, 5, 4)(7, 8), \quad s_1 s_3 = (1, 2)(5, 7, 8, 6) \quad \text{and} \quad s_2 s_3 = (1, 4, 7, 6, 3).$$

In each case, $G := \langle s_1, s_2, s_3 \rangle$ is primitive: for instance, when $n = 6$ or 8 it is doubly transitive, in the latter case because $\langle (s_1 s_2)^2, s_3 \rangle$ fixes 8 and is transitive on the remaining points. Jordan's Theorem, applied to a suitable 3- or 5-cycle, then implies that $G = A_n$. In each case $s_1 s_3$ and $s_2 s_3$ have different orders, so there is no automorphism transposing s_1 and s_2 and inverting (equivalently fixing) s_3 . Thus A_6, A_7 and A_8 are realised in these classes, and the same applies to the classes 3 and 4^σ by Lemma 4.5(c).

For the classes $T = 2^\sigma \text{ex}$ the lower bound $n \geq 8$ given above cannot be improved: a result of Singerman (see Proposition 8.1 in the next section) eliminates the groups A_4, A_5 and A_6 , which are isomorphic to $L_2(q)$ for $q = 3, 4$ (or 5) and 9 respectively, while A_7 can be eliminated as follows. If there is an epimorphism $\Gamma^+ \rightarrow A_7$, with $X \mapsto x$ and $Y \mapsto y$, then the involution y has five cycles, namely two transpositions and three fixed points. In order for x and y to generate a transitive group we therefore need x to have a cycle of length at least 5, so it must be a 5-cycle or a 7-cycle. In the first case, it is easy to see (by drawing permutation diagrams, for instance), that any pair generating a transitive group are inverted by some element of S_7 , giving a forbidden automorphism. If x is a 7-cycle, then without loss we may assume that it is $(1, 2, \dots, 7)$. Replacing x with a suitable power we may also assume that y involves a transposition $(i, i+1)$, and then conjugating x and y by a suitable power of x we may assume that $i = 1$, so that $y = (1, 2)(j, k)$ for some $j < k$ in $\{3, \dots, 7\}$. The ten possibilities for (j, k) lead, in the cases $(3, 4), (3, 7), (4, 5), (4, 6), (5, 6)$ and $(6, 7)$, to a pair x, y inverted in S_7 ; the remaining cases $(3, 5), (3, 6), (4, 7)$ and $(5, 7)$ give a pair preserving one of the two $\langle x \rangle$ -invariant Fano plane geometries on \mathbb{F}_7 , where the lines are the translates of the quadratic residues $\{1, 2, 4\}$ or the non-residues $\{3, 5, 6\}$, so that $\langle x, y \rangle$ is a proper subgroup isomorphic to $L_3(2)$. Thus $A_7 \notin \mathcal{G}(2^\sigma \text{ex})$.

If $T = 5^\sigma$, with $N(T) \cong F_2$, we can realise A_7 by using generators $(1, 2, 3, 4, 5)$ and $(1, 6, 7)(2, 4, 5)$: these generate a transitive group and no maximal subgroup of A_7 contains elements of orders 3, 5 and 7. By inspection there are no forbidden automorphisms. However, any generating pair for $A_6 \cong L_2(9)$, of $A_5 \cong L_2(4)$ or of $A_4 \cong L_2(3)$ are inverted by an automorphism, so these groups cannot be realised. \square

This, together with Theorem 6.3, completes the proof of Theorem 1.3.

8 Realising $L_2(q)$

In this section we complete the proof of Theorem 1.1 by considering the groups $L_2(q)$. We need the following fact, a consequence of Macbeath's work [34] on generators of $L_2(q)$, which was observed by Singerman in proving a result [46, Theorem 3] concerning symmetries of Riemann surfaces:

Proposition 8.1 *If two elements generate $L_2(q)$, there is an automorphism of $L_2(q)$ inverting them both.* \square

The case $T = 1$ of the following theorem is already known as a special case of more extensive results of Nuzhin [39, 42] about finite simple groups (see §9), but for completeness we give a simple proof here:

Theorem 8.2 *The group $L_2(q)$ is the automorphism group of a map in an edge-transitive class T if and only if one of the following holds:*

- $T = 1$, and $q \neq 3, 7$ or 9 ;
- $T = 2^\sigma$ for some σ or $T = 3$, and $q \neq 3$;
- $T = 4^\sigma$ for some σ .

In particular, this situation does not arise if $T = 2^\sigma \text{ex}$ or 5^σ for any σ .

Proof. Since $L_2(2) \cong S_3$, $L_2(3) \cong A_4$, $L_2(4) \cong L_2(5) \cong A_5$ and $L_2(9) \cong A_6$, the cases $q \leq 5$ and $q = 9$ are covered by Theorems 5.4 and 7.2, so we may assume from now on that $q = 7$ or $q \geq 11$.

The types $T = 2^\sigma \text{ex}$ and 5^σ are eliminated by Proposition 8.1, which shows that there are always forbidden automorphisms in these cases.

Now let $T = 1$. We need to find involutions r_0 , r_1 and r_2 generating $G := L_2(q)$, with $(r_0 r_2)^2 = 1$, so that G is a quotient of Γ . Without loss of generality we can take

$$r_1 = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We now look for elements $x = r_0 r_1$ and $z = r_1 r_2$ of G , inverted by r_1 , such that $\langle x, z, r_1 \rangle = G$ and $xz = r_0 r_2$ is an involution. The elements of G inverted by r_1 (other than r_1 itself) are those of the form

$$\pm \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad \text{with} \quad ad - b^2 = 1,$$

so we take two such elements

$$x = \pm \begin{pmatrix} a & b \\ b & d \end{pmatrix} \quad \text{and} \quad z = \pm \begin{pmatrix} a' & b' \\ b' & d' \end{pmatrix},$$

where $ad - b^2 = 1$ and similarly for z . Their product has trace $aa' + 2bb' + dd'$, so for xz to be an involution we require

$$aa' + 2bb' + dd' = 0.$$

To define x we can take $b = 0$ and $d = a^{-1} \neq \pm 1$, so that x is elliptic, of order dividing $(q-1)/h$ where $h = \gcd(q-1, 2)$. In this case we require

$$aa' + dd' = 0,$$

that is,

$$d' = -aa'/d = -a^2a',$$

so for z to exist we need

$$1 = a'd' - b'^2 = -(aa')^2 - b'^2.$$

Thus we need $-1 - (aa')^2$ to be a square in \mathbb{F}_q , so that we can take b' to be a square root. For a given $a \neq 0$, this will be true for all a' if q is even, or for $(q+1)/2$ values of a' if q is odd.

If we take a to be a primitive element of \mathbb{F}_q , then $D := \langle r_0, r_1 \rangle = \langle r_1, x \rangle$ is a dihedral group of order $2(q-1)/h$. It follows from Dickson's description of the subgroups of $L_2(q)$ in [10, Chapter XII] that D is a maximal subgroup of G provided $q = 8$ or $q \geq 13$. It consists of the elements of G of the form

$$\pm \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \quad \text{or} \quad \pm \begin{pmatrix} 0 & * \\ * & 0 \end{pmatrix},$$

so if z does not have this form then $\langle r_0, r_1, r_2 \rangle = G$. For this it is sufficient that $a', b' \neq 0$, that is, $a' \neq 0$ and $(aa')^2 \neq -1$. For a given value of a , this excludes one, two or three values of a' as $q \equiv 3, 0$ or $1 \pmod{4}$, so we can choose a suitable value of a' provided $(q+1)/2 > 3$, which is true since $q > 5$.

This realises G for all $q = 2^e \geq 8$ and all odd $q \geq 13$. In the case $q = 11$, although D is not maximal we can ensure that $\langle r_0, r_1, r_2 \rangle = G$ by choosing a' so that z has order 6; for instance, one can take $a = a' = 2$, so that $d' = 3$ and $b' = \pm 2$, giving the map N46.3 of type $\{5, 6\}$ in Conder's list of non-orientable regular maps [3]. Thus $G \in \mathcal{G}(1)$ when $q = 8$ or $q \geq 11$, and hence $G \in \mathcal{G}(T)$ for $T = 2^\sigma, 3$ or 4^σ , by Lemma 4.4(a).

The groups $L_2(7)$ and $L_2(9) (\cong A_6)$ are not members of $\mathcal{G}(1)$, since Nuzhin has shown that neither group is a quotient of Γ : for instance, a simple counting argument shows that in $L_2(7)$, a Klein four-group $\langle r_0, r_2 \rangle$ and a cyclic group $\langle r_1 \rangle$ of order 2 are always contained in one of the 14 maximal subgroups isomorphic to S_4 ; we outlined an equally simple argument for $L_2(9) (\cong A_6)$ in the proof of Theorem 7.2.

Now let $T = 2$. In $L_2(7)$ let us define the involutions

$$s_1 = \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s_2 = \pm \begin{pmatrix} 0 & 2 \\ 3 & 0 \end{pmatrix} \quad \text{and} \quad s' = \pm \begin{pmatrix} 1 & 3 \\ -3 & -1 \end{pmatrix}.$$

Then $s_1 s_2$ and $s_1 s'$ have trace ± 1 and hence have order 3, while $s_2 s'$ has trace ± 3 and hence has order 4. These involutions generate $L_2(7)$, since no maximal subgroup (isomorphic to S_4 or $C_7 \rtimes C_3$) contains two elements of order 3, such as $s_2 s_1$ and $s_1 s'$, with product of order 4. Lemma 4.5(c) now implies that $L_2(7) \in \mathcal{G}(T)$ for all $T = 2^\sigma$, 3 or 4^σ . \square

This complete the proof of Theorem 1.1.

9 Realising other finite simple groups

More generally, we will now consider which finite simple groups are members of $\mathcal{G}(T)$ for each of the 14 edge-transitive classes T . By Lemma 4.3 the only cyclic group of prime order arising is C_2 , for $T = 1, 2^\sigma, 3$ or 4^σ , so we will restrict our attention to non-abelian finite simple groups. Their classification (see [6, 51]) allows us to deal with them by inspection. The alternating groups have already been dealt with in Theorem 1.1, so we can concentrate mainly on the groups of Lie type and the sporadic simple groups.

Example Let G be a Suzuki group $Sz(2^e)$ or a ‘small’ Ree group $R(3^e)$, for some odd $e > 1$. It is known from work of Nuzhin [39, 42] that G is a quotient of Γ , so $G \in \mathcal{G}(T)$ if $T = 1, 2^\sigma, 3$ or 4^σ by Lemma 4.4(a). It is also known from results of Suzuki [48] and Ree [44] that their groups are respectively quotients of the triangle groups $\Delta(2, 4, 5)$ and $\Delta(2, 3, 7)$, with no automorphism inverting the second generator, so $G \in \mathcal{G}(T)$ if $T = 2^\sigma$ or 5^σ by Lemma 4.4(b). Thus $G \in \mathcal{G}(T)$ for all T .

As a first step towards a more comprehensive approach, we can use results already in the literature to describe those non-abelian finite simple groups which are quotients of each parent group $N(T)$, postponing until later any consideration of forbidden automorphisms. The following statement, depending on the work of many authors, summarises the situation. First let \mathbb{M} denote the set of finite simple groups isomorphic to

$L_3(q), U_3(q), L_4(2^e), U_4(2^e), U_4(3), U_5(2), A_6, A_7, M_{11}, M_{22}, M_{23}$ or McL

for some prime power q or integer $e \geq 1$. Note that \mathbb{M} includes the groups $L_2(7) \cong L_3(2)$, $L_2(9) \cong A_6$, $A_8 \cong L_4(2)$, the symplectic group $S_4(3) \cong U_4(2)$ and the orthogonal group $O_6^-(3) \cong U_4(3)$.

Theorem 9.1 *Let T be one of the classes of edge-transitive maps, and let G be a non-abelian finite simple group. Then G is a quotient of the parent group $N(T)$ if and only if one of the following conditions holds:*

- $T = 1$, and $G \notin \mathbb{M}$;
- $T = 2^\sigma$ for some σ , and $G \not\cong U_3(3)$;
- $T = 2^{\sigma\text{ex}}, 3, 4^\sigma$ or 5^σ for some σ .

Proof. This theorem is a simple consequence of a number of well-known results, as follows.

When $T = 1$ (the set of regular maps), so that $N(T) = \Gamma \cong V_4 * C_2$, we are looking for groups generated by three involutions, two of which commute. In 1980 Mazurov asked in the Kourovka Notebook [28, Problem 7.30] which finite simple groups have this property. This problem was subsequently solved for the alternating groups and the simple groups of Lie type by Nuzhin [39, 40, 41, 42]. In [37] Mazurov has given an elegant, unified and largely computer-free discussion of which sporadic simple groups are quotients of Γ , together with a summary of how mathematicians such as Ershov and Nevmerzheritskaya, Nuzhin and Timofeenko, Abasheev, and Norton earlier dealt with various individual sporadic groups, mainly by using computers. The groups $U_4(3)$ and $U_5(2)$ have been added to the published lists, as a computer search using GAP by Martin Mačaj [33] (confirmed independently by Matan Ziv-Ziv also using GAP, and by Marston Conder using Magma) has shown that they are not quotients of Γ . The result of this work is that the only non-abelian finite simple groups which do not have this property are those in the set \mathbb{M} defined above.

When $T = 2^\sigma$, so that $N(T) \cong C_2 * C_2 * C_2$, we are looking for groups generated by three involutions, with no requirement that two of them should commute. Here Malle, Saxl and Weigel [35], building on earlier work of others such as Dalla Volta (see [9] for the sporadic groups, for instance), have shown that the only non-abelian finite simple group without such generators is the unitary group $U_3(3)$, shown by Wagner [49] to require four involutions. Thus with this one exception, every non-abelian finite simple group is a quotient of $N(2^\sigma)$, whereas they are all quotients of $N(3) \cong C_2 * C_2 * C_2 * C_2$.

When $T = 2^\sigma \text{ex}$, so that $N(T) \cong C_\infty * C_2$, we are looking for groups generated by two elements, one of them an involution. Here it is known (again, see [35]) that every non-abelian finite simple group has such a generating pair, so it is a quotient of $N(T)$. Clearly the same applies when $T = 4^\sigma$, so that $N(T) \cong C_2 * C_2 * C_\infty$, and when $T = 5^\sigma$, so that $N(T) \cong F_2$. \square

Theorem 9.1 immediately shows that the non-abelian finite simple groups $G \in \mathcal{G}(1)$ are those not in \mathbb{M} , since for the class $T = 1$ there are no forbidden automorphisms to consider. Moreover, Lemma 4.4(a) then gives:

Corollary 9.2 *If G is a non-abelian finite simple group not in \mathbb{M} , then $G \in \mathcal{G}(T)$ for each edge-transitive class $T = 1, 2^\sigma, 3$ or 4^σ .* \square

In order to prove Theorem 1.2 we need to determine, for each of the remaining classes class $T \neq 1$, which of the simple quotients G of $N(T)$ have generators admitting no forbidden automorphisms, so that $G \in \mathcal{G}(T)$.

Leemans and Liebeck [29, 30] have recently proved the following theorem in the context of chiral polyhedra, which are equivalent to maps in class 2^Pex . First let \mathbb{L} denote the set of finite simple groups which are isomorphic to

$$L_2(q), L_3(q), U_3(q) \text{ or } A_7$$

for some prime power q .

Theorem 9.3 (Leemans and Liebeck) *A non-abelian finite simple group is generated by elements x and y , where y is an involution and no automorphism inverts x and fixes y , if and only if it is not a member of \mathbb{L} .* \square

The proof that every non-abelian finite simple group $G \notin \mathbb{L}$ has such a generating pair is given in [30]. It is straightforward to show that A_7 and $L_2(q)$ do not have such a pair (see Theorems 7.2 and 8.2, for example); the proof for $L_3(q)$ and $U_3(q)$ is (unsurprisingly) much harder, and the authors of [30] intend to present this separately in [31]. Lemma 4.4(b) immediately implies the following:

Corollary 9.4 *If G is a non-abelian finite simple group which is not in \mathbb{L} , then $G \in \mathcal{G}(T)$ for each edge-transitive class $T = 2^\sigma \text{ex}, 4^\sigma$ or 5^σ .* \square

Combining this with Corollary 9.2, we have:

Corollary 9.5 *If G is a non-abelian finite simple group which is not in $\mathbb{L} \cup \mathbb{M}$, then $G \in \mathcal{G}(T)$ for each edge-transitive class T .* \square

Thus, in order to prove Theorem 1.2 we may restrict attention to the groups $G \in \mathbb{L} \cup \mathbb{M}$, in each case determining those classes T such that $G \in \mathcal{G}(T)$. The groups $L_2(q)$ and A_n have been dealt with in Theorem 1.1, as have $L_3(2) \cong L_2(7)$ and $L_4(2) \cong A_8$, so the following groups remain:

- $L_3(q)$ and $U_3(q)$ for prime powers $q > 2$;
- $L_4(2^e)$ for $e > 1$ and $U_4(2^e)$ for $e \geq 1$;
- $U_4(3)$ and $U_5(2)$;
- M_{11} , M_{22} , M_{23} and McL .

To deal with specific examples of these groups we will use information and notation concerning their conjugacy classes, characters, maximal subgroups, automorphisms, etc in the ATLAS [6], together with the following formula, due to Frobenius [12]:

Proposition 9.6 *Let \mathcal{A} , \mathcal{B} and \mathcal{C} be conjugacy classes in a finite group G . Then the number of solutions of the equation $abc = 1$ in G , with $a \in \mathcal{A}$, $b \in \mathcal{B}$ and $c \in \mathcal{C}$, is*

$$\frac{|\mathcal{A}| \cdot |\mathcal{B}| \cdot |\mathcal{C}|}{|G|} \sum_{\chi} \frac{\chi(a)\chi(b)\chi(c)}{\chi(1)} \quad (2)$$

where χ ranges over the irreducible complex characters of G . \square

Concerning the groups $L_3(q)$ and $U_3(q)$, we have the following corollary to Theorem 9.3:

Corollary 9.7 *If $G = L_3(q)$, or if $G = U_3(q)$ where $q > 3$, then $G \in \mathcal{G}(T)$ for each $T = 2^\sigma$, 3 or 4^σ .*

Proof. Malle, Saxl and Weigel [35] have shown that every non-abelian finite simple group $G \not\cong U_3(3)$ is generated by a strongly real element x (one such that $x^a = x^{-1}$ for some involution $a \in G$) and an involution y . Thus G is generated by three involutions a , $b := ax$ and y . If an automorphism of G fixes the involution y and transposes a and b , then it inverts x . However,

by Theorem 9.3 the groups $G = L_3(q)$ and $U_3(q)$ have no generating pairs x, y with this property, so by taking $s_1 = a$, $s_2 = b$ and $s_3 = y$ we see that $G \in \mathcal{G}(T)$ for $T = 2$, and hence for $T = 2^\sigma$, 3 or 4^σ by Lemma 4.5(c). \square

The group $G = U_3(3)$ was excluded from Corollary 9.7, so we treat it here as a special case:

Theorem 9.8 *The group $U_3(3)$ is in $\mathcal{G}(T)$ for each $T = 3, 4^\sigma$ or 5^σ .*

Proof. First let $T = 3$. The group $G = U_3(3)$ has a single conjugacy class of maximal subgroups $H \cong L_2(7)$. Let s_1, s_2 and s_3 be involutions in such a subgroup H corresponding to the elements

$$\pm \begin{pmatrix} -1 & 1 \\ -2 & 1 \end{pmatrix}, \quad \pm \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \pm \begin{pmatrix} 0 & 3 \\ 2 & 0 \end{pmatrix}$$

of $L_2(7)$, so that s_1s_2 , s_1s_3 and s_2s_3 have orders 4, 4 and 3 respectively. Since $s := s_1s_2s_3$ has order 7, and no proper subgroup of $L_2(7)$ has order divisible by 14, these involutions generate H .

Now G acts as a primitive group of degree 36 and rank 4 on the cosets of H , with subdegrees 1, 7, 7 and 21. The subgroup $\langle s_1, s_2 \rangle \cong D_4$ of H thus fixes more than one point, so it lies in a second subgroup $H' \cong L_2(7)$ of G . The subgroups of $L_2(7)$ isomorphic to D_4 are all conjugate (they are Sylow 2-subgroups), so as above there is an involution $s_0 \in H'$ such that $\langle s_1, s_2, s_0 \rangle = H'$ where s_1s_2 , s_1s_0 and s_2s_0 have orders 4, 4 and 3. Then $\langle s_0, s_1, s_2, s_3 \rangle = \langle H, H' \rangle = G$ and the partitions $02 \mid 13$ and $01 \mid 23$ satisfy the condition in Lemma 4.6, so $G \in \mathcal{G}(3)$.

Next, let $T = 4$. The involution s_1 and the element s of order 7 defined above generate the maximal subgroup H of G . We need to show that there is another involution $s'_2 \in G \setminus H$ such that no automorphism of G inverts s and transposes s_1 and s'_2 . There are seven automorphisms inverting s , all of them outer automorphisms in the conjugacy class $2B \subset \text{Aut } G = PGU_3(3)$, forming a coset of $C_G(s) = \langle s \rangle$. Now there are 63 involutions in G , of which 21 are in H . We can therefore choose one of the 42 involutions $s'_2 \in G \setminus H$, avoiding the seven images of s_1 under the automorphisms inverting s . Thus $G \in \mathcal{G}(4^\sigma)$ for all σ .

Finally let $T = 5$. There are no maximal subgroups of G containing elements s and s' of orders 7 and 6, so any such pair generates G . Since they have different orders, it is sufficient to show that there exists such a pair

which are not simultaneously inverted by any automorphism. Any element s of order 7 is inverted by seven automorphisms, all outer, in class $2B$. Each such automorphism inverts 24 elements $s' \in G$ of order 6, so there are at most 168 such elements s' inverted by automorphisms also inverting s . Since G has 504 elements of order 6, a suitable pair s, s' exists, so $G \in \mathcal{G}(5^\sigma)$ for all σ . \square

For arguments involving Singer cycles and their eigenvalues, we will need the following result:

Lemma 9.9 *Let λ be a primitive root in a finite field \mathbb{F} . If λ is conjugate to λ^{-1} under $\text{Gal } \mathbb{F}$ then $|\mathbb{F}| \leq 4$.*

Proof. Let $|\mathbb{F}| = p^e$ where p is prime, so that $\text{Gal } \mathbb{F}$ is generated by the Frobenius automorphism $t \mapsto t^p$, which has order e . If λ is conjugate to its inverse, then $\lambda^{p^f} = \lambda^{-1}$ and hence $p^f + 1$ is divisible by the multiplicative order $p^e - 1$ of λ , for some $f = 0, 1, \dots, e - 1$. Hence $p^e - 1 \leq p^{e-1} + 1$, so $p^{e-1}(p - 1) \leq 2$, and the result follows immediately. \square

Theorem 9.10 *If $q > 2$ then $L_3(q), U_3(q) \in \mathcal{G}(5^\sigma)$ for all σ .*

Proof. Let $G = L_3(q)$, of order $q^3(q^3 - 1)(q^2 - 1)/d$ where $d = (3, q - 1)$. Then $\text{Aut } G$ is a semidirect product of $P\Gamma L_3(q)$ by a group of order 2 generated by the graph automorphism (or polarity) γ induced by the matrix operation $A \mapsto (A^{-1})^T$. Let s be a Singer cycle in G , that is, an element of order $(q^2 + q + 1)/d$; its centraliser in G is $\langle s \rangle$, while its centraliser in $\text{Aut } G$ is a cyclic group C of order $q^2 + q + 1$; this is generated by a Singer cycle in $PGL_3(q)$, which contains G with index d . It follows by applying results of Berczky [1] to $SL_3(q)$ that the only maximal subgroup of G containing s is $M := N_G(\langle s \rangle)$, a semidirect product of $\langle s \rangle$ by C_3 .

Simple numerical estimates show that if $q > 2$ there exist elements $s' \in G$ such that $s' \notin M$ (so $\langle s, s' \rangle = G$), s' is not a Singer cycle (so no automorphism can transpose it with $s^{\pm 1}$) and none of the $q^2 + q + 1$ automorphisms inverting s also inverts s' . (It follows from Lemma 9.9 that the automorphisms inverting s are all conjugate in $\text{Aut } G$ to γ ; the elements of G inverted by γ are those corresponding to symmetric or skew-symmetric matrices.) Thus $G \in \mathcal{G}(5^\sigma)$ for all σ .

A similar argument can be applied to the groups $G = U_3(q)$; these have order $q^3(q^3 + 1)(q - 1)^2/d$ where $d = (3, q + 1)$, and are simple for each prime

power $q > 2$. In this case $\text{Aut } G = \text{PTU}_3(q)$; this is an extension of $\text{PGU}_3(q)$, which contains G with index d , by $\text{Gal } \mathbb{F}_{q^2}$. The analogue in G of a Singer cycle in $L_3(q)$ is an element s of order $(q^2 - q + 1)/d$; its centraliser in G is $\langle s \rangle$, while its centraliser in $\text{Aut } G$ has order $q^2 - q + 1$. In each case s is contained in a unique maximal subgroup M of G : if $q \neq 3$ or 5 then $M = N_G(\langle s \rangle)$, a semidirect product of $\langle s \rangle$ by C_3 , whereas if $q = 3$ or 5 then $M \cong L_2(7)$ or A_7 . There is a unique conjugacy class \mathcal{A} of involutions $\alpha \in \text{Aut } G$ inverting such elements s , namely the class (denoted by $2B$ for the groups in [6]) containing the automorphism γ induced by the automorphism $t \mapsto t^q$ of \mathbb{F}_{q^2} . Each s is inverted by $q^2 - q + 1$ automorphisms $\alpha \in \mathcal{A}$. Each $\alpha \in \mathcal{A}$ inverts an element $s' \in G$ if and only if $s'\alpha \in \mathcal{A}$, so the number of elements s' inverted by α is $|\mathcal{A}| = q^2(q - 1)(q^2 - q + 1)$. As before, it follows that there exist such pairs s, s' which generate G with no forbidden automorphisms. \square

Theorems 9.1 and 9.3 show that the groups $L_3(q)$ and $U_3(q)$ are not in $\mathcal{G}(T)$ for $T = 1$ or 2^σex , so this completes the proof of Theorem 1.2 for these two families of simple groups. Theorems 9.1 and 9.3 also show that the groups $L_4(2^e)$ and $U_4(2^e)$ are not in $\mathcal{G}(1)$, but that they are in $\mathcal{G}(T)$ for $T = 2^\sigma \text{ex}$, and hence by Lemma 4.4(b) also for $T = 4^\sigma$ and 5^σ . To complete the proof of Theorem 1.2 for these groups we prove the following:

Theorem 9.11 *The groups $L_4(2^e)$ and $U_4(2^e)$ are in $\mathcal{G}(2^\sigma)$ and $\mathcal{G}(3)$ for all σ and all $e \geq 1$, as are $U_4(3)$ and $U_5(2)$.*

Proof. Let $G = L_4(q)$ where $q = 2^e$. We may assume that $e > 1$, since the group $L_4(2) \cong A_8$ has been dealt with in Theorem 7.2. In [35, Theorem 2.1], Malle, Saxl and Weigel showed that there are conjugacy classes $\mathcal{C}_1, \mathcal{C}_2 \subset G$, containing elements of orders $q^2 + 1$ and $q^3 - 1$ respectively, such that the elements of \mathcal{C}_1 are strongly real, and for each non-identity conjugacy class $\mathcal{C}_3 \subset G$ there are elements $g_i \in \mathcal{C}_i$ ($i = 1, 2, 3$) generating G and satisfying $g_1 g_2 g_3 = 1$. We can write $g_1 = s_2 s_3$ for involutions $s_2, s_3 \in G$. There are two conjugacy classes of involutions in G , of different sizes, depending on the dimensions of the subspaces of the natural module \mathbb{F}_q^4 fixed by their elements. The involutions inverting g_1 , such as s_2 and s_3 , all lie in one of these two classes, so taking \mathcal{C}_3 to be the other class of involutions, and putting $s_1 = g_3$, we obtain three involutions s_1, s_2, s_3 generating G , with no automorphism transposing s_1 and s_2 . This shows that $G \in \mathcal{G}(2)$, so Lemma 4.5(c) implies that $G \in \mathcal{G}(T)$ for $T = 2^\sigma$ and for $T = 3$.

The same argument can be applied to the groups $G = U_4(q)$ for $q = 2^e$, using [35, Theorem 2.2], the only difference being that the elements $g_2 \in \mathcal{C}_2$ now have order $q^3 + 1$ rather than $q^3 - 1$.

In the case of $U_5(2)$, classes \mathcal{C}_1 and \mathcal{C}_2 in [35, Theorem 2.2] contain elements of orders $(q^5 + 1)/d(q + 1) = 11$ and $q^{4/2} + (-1)^{4/2} = 5$ (where $d = (5, q + 1) = 1$), the latter strongly real. There are two conjugacy classes of involutions, and if we choose \mathcal{C}_3 to be the class $2A$ of involutions not inverting elements of order 5, the proof proceeds as before.

The group $U_4(3)$, which has only one class of involutions, resists this line of argument; however, a computer search by Matan Ziv-Av [53], using GAP, has shown that it satisfies the conditions of Lemma 4.5(a), for example with ab , ac and bc having orders 4, 5 and 6, so Lemma 4.5(c) applies. \square

Finally, to complete the proof of Theorem 1.2 we deal with the sporadic simple groups in $\mathbb{L} \cup \mathbb{M}$:

Theorem 9.12 *The Mathieu groups M_{11} , M_{22} , M_{23} and the McLaughlin group McL are in $\mathcal{G}(T)$ for all edge-transitive classes $T \neq 1$, but they are not in $\mathcal{G}(1)$.*

Proof. Theorem 9.1 shows that these groups are not in $\mathcal{G}(1)$. Theorem 9.3 shows that they are in $\mathcal{G}(2^P \text{ex})$, so by Lemma 4.4(b) they are in $\mathcal{G}(T)$ for all classes $T = 2^\sigma \text{ex}$, 4^σ and 5^σ .

For $T = 2^\sigma$ or 3 we use Lemma 4.5(b) and (c). The group $G = M_{11}$ has unique conjugacy classes $2A$ and $4A$ of elements of order 2 and 4, and two mutually inverse classes $11A$ and $11B$ of elements of order 11. Proposition 9.6 and the character table of G in [6] show that there are elements $a, b, c \in G$, in classes $2A$, $4A$ and $11A$, such that $abc = 1$. The maximal subgroups of G are listed in [6], and none contains elements of orders 4 and 11, so $\langle a, b \rangle = G$. A second application of Proposition 9.6, or equivalently the existence of subgroups isomorphic to D_4 , shows that there are involutions $s_1, s_2 \in G$ with $s_1 s_2 = b$. Since G has no outer automorphisms, no automorphism can invert c , so Lemma 4.5 shows that $G \in \mathcal{G}(T)$ for all classes $T = 2^\sigma$ and 3.

Essentially the same argument gives the result for M_{23} , but now with a, b and c in classes $2A$, $8A$ and $23A$, and also for McL , using classes $2A$, $12A$ and $11A$. (Even though McL has outer automorphisms, they do not transpose the mutually inverse classes $11A$ and $11B$.)

A similar argument can be applied to $G = M_{22}$, but in this case the details are less straightforward. This group has one class $2A$ of involutions,

one class $6A$ of elements of order 6, and two mutually inverse classes $7A$ and $7B$ of elements of order 7; the latter are not transposed in $\text{Aut } G$, which contains G with index 2. Proposition 9.6 shows that G contains $12|G|$ triples (a, b, c) of elements of orders 2, 6 and 7 with $abc = 1$. By inspection of the list of maximal subgroups of G , each of these triples can generate one of the $2|G : H|$ subgroups $H \cong A_7$, or one of the $|G : H|$ subgroups $H \cong \text{AGL}_3(2)$, or G itself. The uniqueness of the corresponding permutation diagram for a and b shows that A_7 is generated by $|S_7| = 2|A_7|$ such triples, giving a total of $4|G|$ triples (a, b, c) generating subgroups $H \cong A_7$. The affine group $\text{AGL}_3(2)$ is a semidirect product $V_8 \rtimes \text{GL}_3(2)$, and triples (a, b, c) generating this group map onto triples $(\bar{a}, \bar{b}, \bar{c})$ of type $(2, 3, 7)$ generating $\text{GL}_3(2) = L_3(2)$. There are $2|L_3(2)|$ such triples, each lifting to 16 triples (a, b, c) , giving a total of $|G : H| \cdot 16 \cdot 2|L_3(2)| = 4|G|$ triples (a, b, c) generating subgroups $H \cong \text{AGL}_3(2)$. This leaves $4|G|$ triples (a, b, c) generating G (and corresponding to a unique chiral pair of orientably regular maps of type $\{6, 7\}$ with automorphism group G). The element $b \in 6A$ in such a triple is strongly real (since A_7 , and hence G , contains subgroups isomorphic to D_6), and the outer automorphisms of G do not transpose the classes $7A$ and $7B$ containing c and c^{-1} , so Lemma 4.5 applies as before. \square

10 Realising nilpotent and solvable groups

Apart from a few small examples, all the groups we have considered so far as candidates for automorphism groups have been non-solvable. In this section we will consider nilpotent and solvable groups, proving Theorem 1.4.

First we need some examples of nilpotent groups. For each prime power $n = p^e$ and each integer $f = 1, 2, \dots, e$, let

$$G = G_{p,e,f} = \langle g, h \mid g^n = h^n = 1, h^g = h^{p^f+1} \rangle, \quad (3)$$

a semidirect product of a normal subgroup $\langle h \rangle \cong C_n$ by $\langle g \rangle \cong C_n$. As a finite p -group, G is nilpotent. (These groups were studied in connection with embeddings of complete graphs $K_{n,n}$ for $p > 2$ in [24], and for $p = 2$ in [11].)

Theorem 10.1 *Let T be an edge-transitive class.*

- *If $T = 1, 2^\sigma, 3$ or 4^σ for some σ , then $\mathcal{G}(T)$ contains finite nilpotent groups of class c for each $c \geq 1$.*

- If $T = 2^\sigma \text{ex}$ for some σ , then $\mathcal{G}(T)$ contains finite nilpotent groups of class c if and only if $c \geq 5$.
- If $T = 5^\sigma$ for some σ , then $\mathcal{G}(T)$ contains finite nilpotent groups of class c if and only if $c \geq 2$.

Proof. Lemma 4.3 deals with abelian automorphism groups, so we may restrict attention to groups of class $c \geq 2$. If $m = 2^e$ then the regular map $\{m, 2\}$, embedding a circuit of length m in the sphere, has automorphism group $D_m \times C_2$, which is nilpotent of class $c = e$. This proves the result for $T = 1$, and hence, by Lemma 4.4(a), for $T = 2^\sigma$, 3 and 4^σ .

Next let $T = 5$. For any odd prime power p^e put $f = 1$ in (3) and define

$$G = G_{p,e,1} = \langle g, h \mid g^{p^e} = h^{p^e} = 1, h^g = h^{p+1} \rangle.$$

As shown in [24], G has centre $Z = \langle g^{p^{e-1}}, h^{p^{e-1}} \rangle \cong C_p \times C_p$, with $G/Z \cong G_{p,e-1,1}$ if $e > 1$, so induction on e shows that G has class $c = e$. There is an epimorphism $N(5) = F_2 \rightarrow G$ given by $S \mapsto s := g$ and $S' \mapsto s' := h$. It is straightforward to check that if $e > 1$ then G has no forbidden automorphisms, so $G \in \mathcal{G}(5^\sigma)$ for each σ .

Finally, let $T = 2^P \text{ex}$. In [36], Malnič, Nedela and Škoviera classified the orientably regular maps with nilpotent automorphism groups of class $c = 2$, and none of them are chiral. This was extended to the cases $c = 3$ and 4 by Conder, Du, Nedela and Škoviera in [4], using a computer search.

On the other hand Du, Kwak, Nedela, Škoviera and the author showed in [11] that if $n = 2^e \geq 4$ then for each $f = 2, \dots, e$ there are orientably regular embeddings \mathcal{M} of the complete bipartite graph $K_{n,n}$ such that the subgroup $G := \text{Aut}_0^+ \mathcal{M}$ of $\text{Aut} \mathcal{M}$ preserving orientation and vertex-colours is isomorphic to the group $G_{2,e,f}$ defined in (3); moreover, if $f \leq e-2$ (so that $e \geq 4$) these maps are chiral, and thus in class T , with $A := \text{Aut} \mathcal{M} \cong G \rtimes C_2$. These maps include examples where $f = 2$,

$$A = \langle g, h, \alpha \mid g^{2^e} = h^{2^e} = \alpha^2 = 1, h^g = h^5, g^\alpha = gh, h^\alpha = h^{-1} \rangle,$$

and the epimorphism $N(T) = \Gamma^+ \rightarrow A$ is given by $X \mapsto x := g, Y \mapsto y := \alpha$. As a finite 2-group, A is nilpotent. We will compute its nilpotence class c as the length of its upper central series

$$1 = Z_0 < Z_1 < \dots < Z_c = A,$$

where Z_i/Z_{i-1} is the centre $Z(A/Z_{i-1})$ of A/Z_{i-1} for each $i \geq 1$.

First note that $Z_1 := Z(A) \leq Z(G)$, since each element $a \in A \setminus G$ has the form $a = \alpha g^i h^j$, giving $h^a = h^{-5^i}$ with $-5^i \equiv -1 \pmod{4}$, so that $h^a \neq h$ since $h^2 \neq 1$. Thus Z_1 is the subgroup of $Z(G)$ commuting with α .

Each element of G has the unique form $g^i h^j$ with $i, j \in \mathbb{Z}_{2^e}$. Multiplication in G is given by

$$g^i h^j \cdot g^k h^l = g^{i+k} h^{5^k j + l},$$

from which it follows, as shown in [11], that G has centre

$$Z(G) = \langle u := g^{2^{e-2}}, v := h^{2^{e-2}} \rangle \cong C_4 \times C_4.$$

We need to find the action of α on $Z(G)$. Since $h^\alpha = h^{-1}$ we have $v^\alpha = v^{-1}$. We also need $u^\alpha = (g^{2^{e-2}})^\alpha = (gh)^{2^{e-2}}$. Induction on e gives

$$(gh)^{2^{e-2}} = g^{2^{e-2}} h^s \quad \text{where} \quad s = 5^{2^{e-2}-1} + 5^{2^{e-2}-2} + \dots + 5 + 1.$$

Lemma 10.2 *If $e \geq 3$ then*

$$5^{2^{e-2}-1} + 5^{2^{e-2}-2} + \dots + 5 + 1 \equiv -2^{e-2} \pmod{2^e}.$$

Proof. Use induction on e , noting that if s_e denotes the left-hand side, then $s_e = (5^{2^{e-2}} + 1)s_{e-1}$ with $5^{2^{e-3}} \equiv 2^{e-1} + 1 \pmod{2^e}$. \square

Thus $u^\alpha = g^{2^{e-2}} h^{-2^{e-2}} = uv^{-1}$. It follows that α conjugates $u^i v^j$ to $u^i v^{-i-j}$, so it centralises this element if and only if $i \equiv -2j \pmod{4}$. Thus

$$Z_1 = \langle u^{-2}v \rangle \cong C_4,$$

so the central quotient $\overline{A} := A/Z_1$ has a presentation Π_e of the form

$$\langle g, h, \alpha \mid g^{2^e} = \alpha^2 = 1, h^{2^{e-2}} = g^{2^{e-1}}, h^g = h^5, g^\alpha = gh, h^\alpha = h^{-1} \rangle.$$

As before, $Z(\overline{A}) \leq Z(\overline{G})$ where $\overline{G} := G/Z_1$. Each element of \overline{G} has the unique form $g^i h^j$ where $i \in \mathbb{Z}_{2^e}$ and $j = 0, 1, \dots, 2^{e-2} - 1$. Multiplication and the action of α are as above for G , but now we use $h^{2^{e-2}} = g^{2^{e-1}}$ to reduce elements to standard form. Calculations similar to those for G show that

$$Z(\overline{G}) = \langle u := g^{2^{e-3}}, v := h^{2^{e-3}} \rangle = \langle u \rangle \times \langle u^{-2}v \rangle \cong C_8 \times C_2,$$

where u and v (now redefined) have orders 8 and 4, with $u^4 = v^2$. As before, we have $v^\alpha = v^{-1}$ and $u^\alpha = (gh)^{2^{e-3}} = g^{2^{e-3}} h^{-2^{e-3}} = uv^{-1}$, giving

$$Z_2/Z_1 = Z(\overline{A}) = \langle u^4, u^{-2}v \rangle \cong V_4.$$

The central quotient $\overline{\overline{A}} := \overline{A}/Z(\overline{A}) = A/Z_2$ therefore has a presentation

$$\langle g, h, \alpha \mid g^{2^{e-1}} = \alpha^2 = 1, h^{2^{e-3}} = g^{2^{e-2}}, h^g = h^5, g^\alpha = gh, h^\alpha = h^{-1} \rangle.$$

This is the presentation Π_{e-1} , obtained from the presentation Π_e for \overline{A} by replacing e with $e - 1$. It follows that $Z_3/Z_2 = Z(\overline{\overline{A}}) \cong V_4$, and all further quotients Z_i/Z_{i-1} in the upper central series of A are also isomorphic to V_4 , until we reach a quotient A/Z_{e-1} of A with presentation Π_2 , namely

$$\langle g, h, \alpha \mid g^4 = \alpha^2 = 1, h = g^2, h^g = h^5, g^\alpha = gh, h^\alpha = h^{-1} \rangle \cong D_4.$$

This has centre $\langle g^2 \rangle \cong C_2$, with central quotient V_4 . Thus the successive quotients in the upper central series of A are isomorphic to

$$C_4, V_4, V_4, \dots, V_4 \text{ (} e - 2 \text{ times)}, C_2 \text{ and } V_4,$$

so that A has class $c = e + 1$. This shows that the maps \mathcal{M} defined above realise nilpotent groups of all classes $c \geq 5$, as required. \square

Example When $T = 2^P \text{ex}$, taking $e = 4$ in the above proof gives a chiral pair of maps of type $\{32, 16\}$ and genus 105, as shown in [11]; these are the duals of the maps C105.25 in [3], with $|A| = 512$ and $c = 5$. (However, according to [4] the smallest chiral pair realising a nilpotent group are the duals of C25.1, of type $\{16, 4\}$ and genus 25, with $|A| = 256$ and $c = 6$.)

The automorphism groups constructed in the proof of Theorem 10.1 have bounded derived length, since D_m and $G_{p,e,f}$ are metabelian. By contrast, we have the following result:

Theorem 10.3 *Let T be an edge-transitive class. Then $\mathcal{G}(T)$ contains a finite solvable group of derived length l if and only if either*

- $T = 1, 2^\sigma, 3$ or 4^σ for some σ and $l \geq 1$, or
- $T = 2^\sigma \text{ex}$ or 5^σ for some σ and $l \geq 2$.

Proof. Since Lemma 4.3 deals with abelian groups, we may assume that $l \geq 2$. First let $T = 1$, so that $N(T) = \Gamma \cong V_4 * C_2$. Define a sequence of normal (in fact, characteristic) subgroups Ψ_n of Γ by

$$\Psi_0 = \Gamma, \quad \text{and} \quad \Psi_{n+1} = \Psi'_n \Psi_n^2 \quad \text{for} \quad n \geq 0.$$

Thus Ψ_n/Ψ_{n+1} is the maximal quotient of Ψ_n which is an elementary abelian 2-group, so Ψ_n is a free group of rank r_n , and $|\Psi_n/\Psi_{n+1}| = 2^{r_n}$, where

$$r_1 = 3 \quad \text{and} \quad r_{n+1} = 2^{r_n}(r_n - 1) + 1 \quad \text{for} \quad n \geq 1.$$

Each quotient Γ/Ψ_n is a finite solvable group of derived length $l = n$: clearly $l \leq n$, and one can prove equality by induction on n , using the fact that each two-step quotient Ψ_{n-1}/Ψ_{n+1} is non-abelian, a consequence of the existence of finitely generated non-abelian groups of exponent 4. The maps \mathcal{M}_n corresponding to the subgroups Ψ_n prove the result for the class $T = 1$, and the result follows for the classes $T = 2^\sigma$, 3 and 4^σ by Lemma 4.4(a) since Γ/Ψ_n is non-abelian for $n \geq 2$.

Now let $T = 2^P\text{ex}$, so that $N(T) = \Gamma^+$. We can argue as before, starting instead with $\Psi_0 = \Gamma^+$, so that subsequent terms Ψ_n ($n \geq 1$) are identical to those used above. The corresponding maps \mathcal{M}_n cannot be used in this case, since they are regular. Instead, we can replace them with their joins $\mathcal{N}_n := \mathcal{M}_n \vee \mathcal{S}$, corresponding to normal subgroups $N_n := \Psi_n \cap M$ of Γ^+ , where \mathcal{S} is one of the chiral pair of Edmonds embeddings of the complete graph K_8 , with a metabelian automorphism group $\Gamma^+/M \cong \text{AGL}_1(8) \cong V_8 \rtimes C_7$ (see [19]). Since $\text{AGL}_1(8)$ has no subgroups of index 2 we have $\Psi_n M = \Gamma^+$, so

$$\Gamma^+/N_n = \Psi_n/N_n \times M/N_n \cong \Gamma^+/M \times \Gamma^+/\Psi_n.$$

Here both direct factors are characteristic subgroups (the first is generated by the elements of odd order, and the second is its centraliser), so a forbidden automorphism of Γ^+/N_n would induce one on Γ^+/M , contradicting the chirality of \mathcal{S} . Thus \mathcal{N}_n is in class T , with $\text{Aut } \mathcal{N}_n \cong \Gamma^+/N_n$, a finite solvable group of derived length $l = \max\{n, 2\}$. This deals with the class 2^Pex and hence, by Lemma 4.4(b), with the remaining classes 2^σex and 5^σ . \square

This completes the proof of Theorem 1.4.

11 Infinite automorphism groups

A map \mathcal{M} is compact if and only if the corresponding map subgroups have finite index in Γ . In the case of edge-transitive maps, this is equivalent to $\text{Aut } \mathcal{M}$ being finite, as we have assumed until now. If we remove this restriction, then there is an even greater abundance and variety of maps and automorphism groups in each edge-transitive class T . So much so, in

fact, that it is impossible to give as comprehensive an analysis as that given earlier for compact maps, so instead we shall just illustrate a few phenomena which distinguish non-compact edge-transitive maps from compact ones.. This possibility can be avoided by choosing \tilde{M} so that

11.1 Uncountably many groups and maps

Since Γ is finitely generated, there are only countably many isomorphism classes of compact maps. On the other hand, there are uncountably many in each edge-transitive class. The proof of the following theorem is adapted from Bernhard Neumann's proof [38] that there are uncountably many 2-generator groups (see also [18, §III.B]).

Theorem 11.1 *Each of the 14 classes T of edge-transitive maps contains 2^{\aleph_0} maps \mathcal{M} with empty boundary and with mutually non-isomorphic automorphism groups $\text{Aut } \mathcal{M}$.*

(Of course, it follows that these maps are also mutually non-isomorphic.)

Proof. First let $T = 1$. For each integer $n \geq 5$ such that $n \equiv 1 \pmod{4}$ we define an epimorphism $\theta_n : \Gamma \rightarrow A_n$, $R_i \mapsto r_i$ as follows. We number the vertices of a regular n -gon $1, 2, \dots, n$ in cyclic order, and let

$$r_1 = (1, n)(2, n-1) \dots (m-1, m+1)(m),$$

$$r_2 = (1)(2, n)(3, n-1) \dots (m, m+1)$$

be the reflections fixing the vertices $m := (n+1)/2$ and 1, so that r_1 and r_2 are involutions in A_n , with

$$x := r_1 r_2 = (1, 2, \dots, n).$$

Now let

$$r_0 = (2, 3)(n-1, n),$$

an involution in A_n commuting with r_2 . Then

$$[r_0, x^{-2}] = (1, 2, 3)(4, 5)(n-1, n),$$

so that

$$a := [r_0, x^{-2}]^{-2} = (1, 2, 3).$$

Since x and a generate A_n by Lemma 7.1(c), so do r_0, r_1 and r_2 , and thus θ_n is an epimorphism.

Now let S be any strictly increasing sequence (n_k) of integers satisfying $5 \leq n_k \equiv 1 \pmod{4}$ for all $k \in \mathbb{N}$. The epimorphisms $\theta_{n_k} : \Gamma \rightarrow A_{n_k}$ define an action θ of Γ on the disjoint union Ω of the sets $\Omega_k := \{1, 2, \dots, n_k\}$ for $k \in \mathbb{N}$, with Γ acting as the alternating group $G_k := \text{Alt } \Omega_k \cong A_{n_k}$ on each of its orbits Ω_k . The resulting permutation group $G(S)$ induced by Γ on Ω is thus a quotient of Γ . We will prove the following

Claim: $G(S)$ has a normal subgroup isomorphic to A_n if and only if n is a term n_k in S .

It then follows that the 2^{\aleph_0} sequences S satisfying the above conditions give 2^{\aleph_0} mutually non-isomorphic quotient groups $G(S)$ of Γ , as required.

Proof of the Claim. By the simplicity of the alternating groups G_k , any normal subgroup of $G(S)$ must act on each Ω_k either trivially or as G_k . It follows that any finite normal subgroup N of $G(S)$ must fix all but finitely many orbits Ω_k (otherwise it contains alternating groups of unbounded degrees), so N is a subdirect product of finitely many alternating groups G_k . Since these groups G_k are mutually non-isomorphic simple groups, N is in fact their direct product. It follows that if any alternating group A_n is isomorphic to a normal subgroup of $G(S)$ then n must be a term n_k in S for some k .

Now we prove the converse, that for each $n = n_k$ in S there is a normal subgroup $G_k \cong A_n$ in $G(S)$. For any odd n , the elements $x = (1, 2, \dots, n)$ and $a = (1, 2, 3)$ of A_n satisfy

$$x^{-(i-2)}ax^{i-2} = (i-1, i, i+1) \quad \text{for all } i \in \mathbb{N},$$

where we replace entries with their remainders mod (n) if necessary. It follows that

$$[a, x^{-(i-2)}ax^{i-2}] = 1 \quad \text{if } n \geq i+1 \geq 6,$$

but

$$[a, x^{-(i-2)}ax^{i-2}] \neq 1 \quad \text{if } n = i.$$

The element $X = R_1R_2$ of Γ induces x on each orbit Ω_k , so the element $A := [R_0, X^2]^{-2}$ induces $a = (1, 2, 3)$. It follows that for each $k \in \mathbb{N}$ the element

$$C_k := [A, X^{-(n_k-2)}AX^{n_k-2}]$$

of Γ fixes Ω_l for all $l > k$, but acts non-trivially on Ω_k (and possibly on some orbits Ω_l with $l < k$). The same applies to all conjugates of C_k in Γ , and

hence to its normal closure N in Γ . Acting on Ω , this induces a finite normal subgroup N_k of $G(S)$, specifically the direct product of G_k and possibly some subgroups G_l with $l < k$. Since these direct factors are mutually non-isomorphic simple groups, G_k is a characteristic subgroup of N_k , and hence it is a normal subgroup of $G(S)$, isomorphic to A_{n_k} , as claimed.

This proves the Theorem for the class $T = 1$, and the result follows for the classes $T = 2^\sigma$, 3 and 4^σ by Lemma 4.4(a).

Now let $T = 2^P \text{ex}$, so that $N(T) = \Gamma^+$. The method of proof is similar to that used for $T = 1$, but now we map the generators X and Y of Γ^+ to the elements

$$x := (1, 2, \dots, n) \quad \text{and} \quad y := (1, 2)(3, 4)(5, 6)(8, 9)$$

of A_n for each odd $n \geq 11$. Then

$$[x, y] = (1, 2, 4, 6, 7, 5, 3)(8, 9, 10),$$

so

$$[x, y]^7 = (8, 9, 10).$$

Thus $\langle x, y \rangle = A_n$ by Lemma 7.1(c), and we have an epimorphism $\Gamma^+ \rightarrow A_n$. The proof now proceeds as in the case $T = 1$, again using the elements x and

$$a := x^7[x, y]^7x^{-7} = (1, 2, 3)$$

to prove the Claim, but this time for any strictly increasing sequence S of odd integers $n = n_k \geq 11$. Since $n > 9$, no permutation in S_n inverting x centralises y , so these quotients A_n of Γ^+ have no forbidden automorphisms; the same therefore applies to the quotients $G(S)$ of Γ^+ , since any forbidden automorphism of them would induce forbidden automorphisms of the characteristic subgroups G_k . This deals with the class $T = 2^P \text{ex}$, and Lemma 4.4(b) then extends the result to the remaining classes. \square

Remark There is a shorter but less explicit proof for $T = 1$ (and hence, by Lemma 4.4(a), for $T = 2^\sigma$, 3 and 4^σ). A group H is *SQ-universal* if every countable group can be embedded in a quotient of H . Britton and Levin proved that any free product $A * B$ is SQ-universal, provided $|A| \geq 2$ and $|B| \geq 3$ (see [32, Theorem V.10.3]), so in particular the parent groups $N(T)$ all have this property. Now any finitely generated SQ-universal group H must have uncountably many non-isomorphic quotients: as we have seen,

B. H. Neumann [38] proved that there are uncountably many non-isomorphic finitely generated groups F ; these are all countable, so each can be embedded in some quotient Q of H ; however, Q is countable, so it has only countably many finitely generated subgroups; each isomorphism class of quotients Q can therefore embed only countably many of the groups F , so H must have uncountably many isomorphism classes of quotients. In particular, this applies to the parent groups $H = N(T)$. In the case $T = 1$ there are no forbidden automorphisms to exclude, so the Theorem is proved, but in the cases $T = 2^\sigma$ ex and 5^σ this line of proof breaks down at this point.

11.2 An embedding theorem

The next result shows that for each edge-transitive class T , any phenomenon, no matter how pathological, which can happen within a countable group can happen within a group $G \in \mathcal{G}(T)$. As in the case of Theorem 11.1, SQ-universality provides a quick proof for $T = 1$ (and hence also for $T = 2^\sigma$, 3 and 4^σ), but in order to eliminate forbidden automorphisms a more explicit argument is needed for the remaining classes.

Theorem 11.2 *For each of the 14 edge-transitive classes T , every countable group C is isomorphic to a subgroup of $\text{Aut } \mathcal{M}$ for some map \mathcal{M} in T .*

Proof. Schupp [45, Theorem I], generalising an earlier result of Gorjuškin [13], proved that if A, B and C are groups with $|A| \geq 2$, $|B| \geq 3$ and $|C| \leq |A * B|$, then C can be embedded in a simple group S generated by a pair of subgroups isomorphic to A and B . It follows that for each parent group $N(T)$, every countable group C can be embedded in a simple quotient $S = N(T)/K$ of $N(T)$. Without loss of generality we may assume that $S \not\cong A_9$.

By Theorem 7.2, for each T there is a normal subgroup L of $N(T)$ such that $N(T)/L$ is isomorphic to A_9 and has no forbidden automorphisms. If we take $M := K \cap L$ then $G := N(T)/M$ is a quotient of $N(T)$ isomorphic to $S \times A_9$, so it contains a subgroup isomorphic to C . As non-isomorphic simple groups, S and A_9 are characteristic subgroups of G ; any forbidden automorphisms of G would therefore induce forbidden automorphisms of the quotient A_9 , whereas this quotient has none. Thus M corresponds to a map \mathcal{M} in T with automorphism group G containing a copy of C . \square

Indeed, by arguing as in the Remark following Theorem 11.1, one can use Schupp's result to show that if $T = 1$, 2^σ , 3 or 4^σ then $\mathcal{G}(T)$ contains

uncountably many non-isomorphic simple groups. As before, however, the problem of eliminating forbidden automorphisms means that a separate argument would be needed to prove this for the remaining classes.

11.3 Growth

One can adapt the proof of Theorem 11.2 to produce, for each edge-transitive class T , uncountably many examples of automorphism groups and edge-transitive maps of intermediate growth. In [15] Grigorchuk produced an example of a finitely-generated group which, as he showed later [16], has intermediate growth, in the sense that the number of distinct elements represented by words of length at most l in the generators grows faster than any polynomial in l , but slower than exponentially. In [16] he extended this example by constructing an uncountable set of groups with this property. These groups G are all subgroups of the automorphism group of the infinite rooted binary tree, generated by automorphisms a, b, c and d satisfying

$$a^2 = b^2 = c^2 = d^2 = bcd = 1.$$

Thus $\langle b, c, d \rangle \cong V_4$, so each of these groups G is a quotient of Γ and is therefore the automorphism group of a regular map. Like G , this map has intermediate growth, meaning that the number of vertices, edges or faces at a given graph-theoretic distance from some chosen base point has a rate of growth which is intermediate between polynomial and exponential (see [20]).

Theorem 11.3 *For each class T of edge-transitive maps there are uncountably many maps \mathcal{M} in T such that \mathcal{M} and $\text{Aut } \mathcal{M}$ have intermediate growth.*

Proof. Having intermediate growth is inherited by subgroups of finite index, so each parent group $N(T)$ has uncountably many quotients $S = N(T)/K$ of intermediate growth. As in the proof of Theorem 11.2, by taking $M = K \cap L$ where $N(T)/L$ is isomorphic to A_9 with no forbidden automorphisms, we obtain uncountably many quotients $Q = N(T)/M$ and associated edge-transitive maps \mathcal{M} in T with intermediate growth. To show that Q has no forbidden automorphisms, note that S is residually nilpotent (since each Grigorchuk group induces a finite 2-group on vertices up to any given level in the tree), so $N(T) = KL$, and hence $Q \cong S \times A_9$ with both direct factors as characteristic subgroups. \square

11.4 Decidability

It is reasonable to ask whether there are algorithms to answer ‘sensible’ questions about edge-transitive maps and their automorphism groups. A further result of Schupp [45] eliminates such a hope. A property \mathcal{P} of groups is called a *Markov property* if

- it is preserved by isomorphisms,
- there is a finitely presented group with property \mathcal{P} , and
- there is a finitely presented group which cannot be embedded in any finitely presented group with property \mathcal{P} .

If \mathcal{P} is hereditary (inherited by subgroups) then the last two conditions are equivalent to

- there are finitely presented groups with and without property \mathcal{P} .

Examples of Markov properties include being trivial, finite, nilpotent, solvable, free, and torsion-free.

The edge-transitive maps in each class T have automorphism groups obtained by adding further relations to the standard presentation of $N(T)$. For example, the automorphism group of the icosahedron, isomorphic to $A_5 \times C_2$, is obtained by adding the relations

$$(R_0 R_1)^3 = (R_1 R_2)^5 = 1$$

to the presentation of $\Gamma = N(1)$. When discussing decision problems, it makes sense to restrict attention to finitely presented quotients, obtained by adding finitely many relations in the standard generators to parent groups $N(T)$. The free product decompositions of these groups $N(T)$ imply that the following is an immediate consequence of Schupp’s Theorem II in [45]:

Theorem 11.4 *For each class T of edge-transitive maps, and each Markov property \mathcal{P} of groups, it is undecidable whether or not adding a finite set of relations to the standard presentation of $N(T)$ produces a quotient group with property \mathcal{P} . \square*

For example, it is undecidable whether adding a finite set of relations to those of $N(T)$ yields a map which is compact, or even whether it collapses to the basic map $\mathcal{N}(T)$.

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